

# Probability with Engineering Applications

Fall 1999

Professor Medard

Handouts:

information sheet

problem set #1

copies of slides

# Outline

- Goal-setting or “Why I am taking this class?”
- Random variables
- Counting things

# What will I get out of this course?

- A toolkit and the ability to use it properly
- A new way of thinking about problems
- Probability is fundamental to:
  - networks and communications
  - manufacturing systems
  - design and testing
  - finance and economics.

# What are the goals of ECE 313?

- We have 3 sets of goals:
  - lower level goals: the “mechanics”
  - medium level goals: modeling problems into forms that fit into our frameworks, so we can use our mechanics
  - higher level goals: using the mechanics to guide analysis and design

# Types of problems: medium level goals

- Over a communication channel, what is the probability that I receive a bit in error? Gaussian distributions.
- Given results of a medical test, what should the diagnostic be? Hypothesis testing.
- If I'm trying to determine how many faulty parts come out of an assembly line on average, how many parts should I examine? Law of Large Numbers.
- How many line-cards do I need to ensure my hub goes down on average once every two years? Combinatorics, distributions.

# Types of problems: higher level goals

- If I want to increase robustness to failure of a computer network, how much is it going to cost me?
- How many servers should an ISP purchase to keep delays below a certain level?
- How large should my buffers be if I want to lose a negligible number of packets to buffer overflow?
- From the data of a study, can we tell if a new drug is effective?
- Looking at the costs, payoffs and risks of a project, should I undertake it?
- How should I price a stock?

# **Think - Pair Share**

- Get together with the person next to you
- Come up with a problem which is relevant to your life and career and that you want to be able to tackle at the end of this class

Hold on to that problem - we'll  
look again at it at the end of the  
term!

# How we'll set about achieving our goals

- Lectures:
  - present the mechanics
  - show how we use the mechanics and why we need the mechanics
  - discuss the general applicability of what we learned
- Problem Sets and Tests:
  - make sure we know the mechanics
  - see how to use our mechanics properly
- Reading Assignments:
  - reinforce the mechanics
  - show applicability of what we learned

Where will I achieve my higher  
level goals?

In later classes, throughout your  
career - if you have the tools to  
achieve your goals!

Computer simulations can model probabilistic systems- why do I need to take a class?

**Think - Pair Share**

# Why do I need to take a class?

- You are at a staff meeting - your boss asks you “do you think that can we double the average time between failures of our system without doubling the price of the system”? What do you answer?
- How do *make* the program that tells you the answer?
- Should I program for a whole afternoon to get an answer I could have had in 5 minutes?

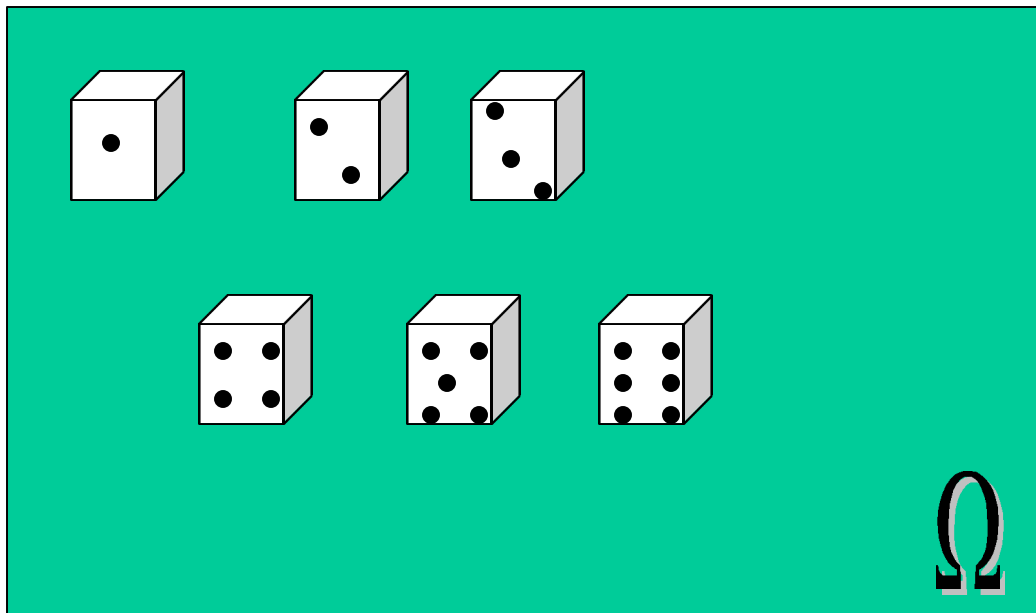
# Outline

- Goal-setting or “Why I am taking this class?”
- Random variables
- Counting things

# What is probability?

- Probability is what we use when **we can't say what is going to happen for sure with the information we have.**
- Balls in the lottery, tossing a coin - system is too complex to model.
- We do not have enough information - what is the probability of rain tonight?
- It's not worth our while to find out exactly - opinion polls.

# Classical construct



Experiment: rolling a die =

**random variable**

Outcome: 5 =

**sample value**

Set or possible outcomes: 1 through 6 =

**sample space  $S$  or  $\Omega$**

# Related concepts

- **Event**: collection of outcomes =
  - I rolled the die and got an even number
  - I rolled the die and got more than 2
- **Null event**: empty set  $\emptyset$ 
  - I rolled the die and got a seven is part of the empty set
- **Sure event or certain event**:  $\Omega$

# Reality Check

- Is a sample value an event?

# Reality Check

- Is a sample value an event?

**YES!**

- Events with one outcome (*singletons*) are called **elementary events**

# Game: Name the Random Variable!

- Whether or not it will rain tomorrow.
- It will rain for half an hour.
- I rolled the die twice and got 2 each time.
- Whether the sum of the outcomes for my first three rolls was 10.
- I had to roll the die at least five times before getting a 4.

# Game: Name the Random Variable!

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# Game: Name the Random Variable!

- It will rain tomorrow for half an hour:
  - whether or not it will rain tomorrow for half an hour (note: exactly half an hour? At least half an hour?)
  - whether or not it will rain tomorrow
  - whether or not it will rain tomorrow for at least 10 minutes
- I rolled the die twice and got 2 each time.
- Whether the sum of the outcomes for my first three rolls was 10.
- I had to roll the die at least five times before getting a 4.

# Game: Name the Random Variable!

- I rolled the die twice and got 2 each time:
  - whether or not I got 2 each time when I rolled the die twice
  - what was the outcome when I rolled the die 5 times (we'll come back to this one)
  - whether or not I got at least 2 even numbers when I rolled the die 4 times.
- Whether the sum of the outcomes for my first three rolls was 10.
- I had to roll the die at least five times before getting a 4.

# Study Self-Test

- Do the last two
  - Whether the sum of the outcomes for my first three rolls was 10.
  - I had to roll the die at least five times before getting a 4.
- Come up with 4 more examples of Name the **Random Variable!**

# Outline

- Goal-setting or “Why I am taking this class?”
- Random variables
- Counting things

# How do I relate **events** to $\Omega$ ?

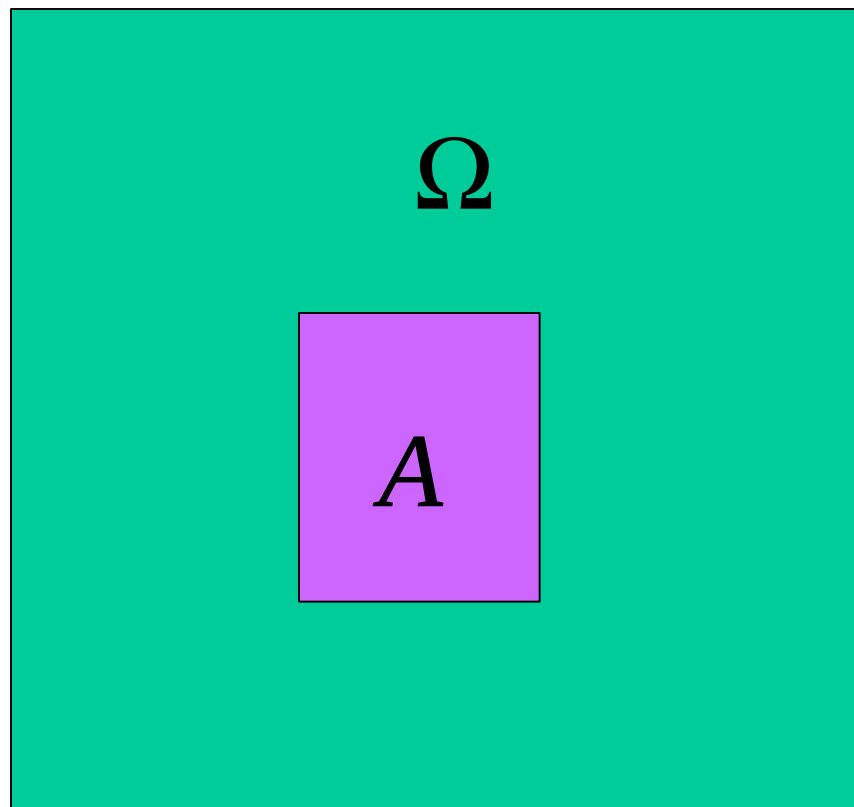
- Random variable: what was the outcome when I rolled the die 5 times
- The sample space  $\Omega$  is the sample space of all possible outcomes from rolling a die 5 times
- An elementary event is (1, 2, 4, 5, 2)  
(*quintuplet - 5 elements in order, shown by parentheses*)

# How do I relate events to $\Omega$ ?

- To relate events to  $\Omega$ , we will relate events to the elementary events of  $\Omega$ ,
- Consider the event A: I had a 1, 2, 4, 5 and then an even number
- A is a set of elementary events:  $A = \{(1, 2, 4, 5, 2), (1, 2, 4, 5, 4), (1, 2, 4, 5, 6)\}$  (the elements of a set are unordered, indicated by  $\{\}$ , ordered sets are indicated by  $()$ )
- How many elementary events are in the event A? Cardinality of A, denoted  $|A|$ .

$$|\Omega| = ?$$

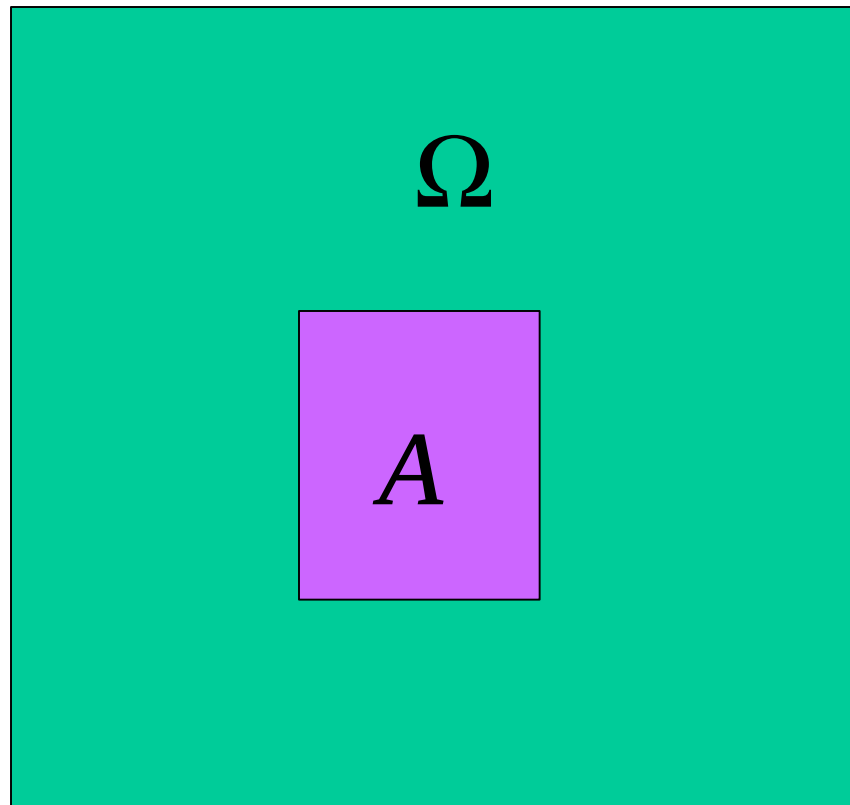
$$|A^c| = ?$$



$$|A| = 3$$

$$|A^c| = 6^5 - 3$$

$$|\Omega| = 6^5$$



$$|A| = 3$$

# Self-Study

- How many elements are there in the set of all 6 rolls of a coin?

# How do I relate events to $\Omega$ ?

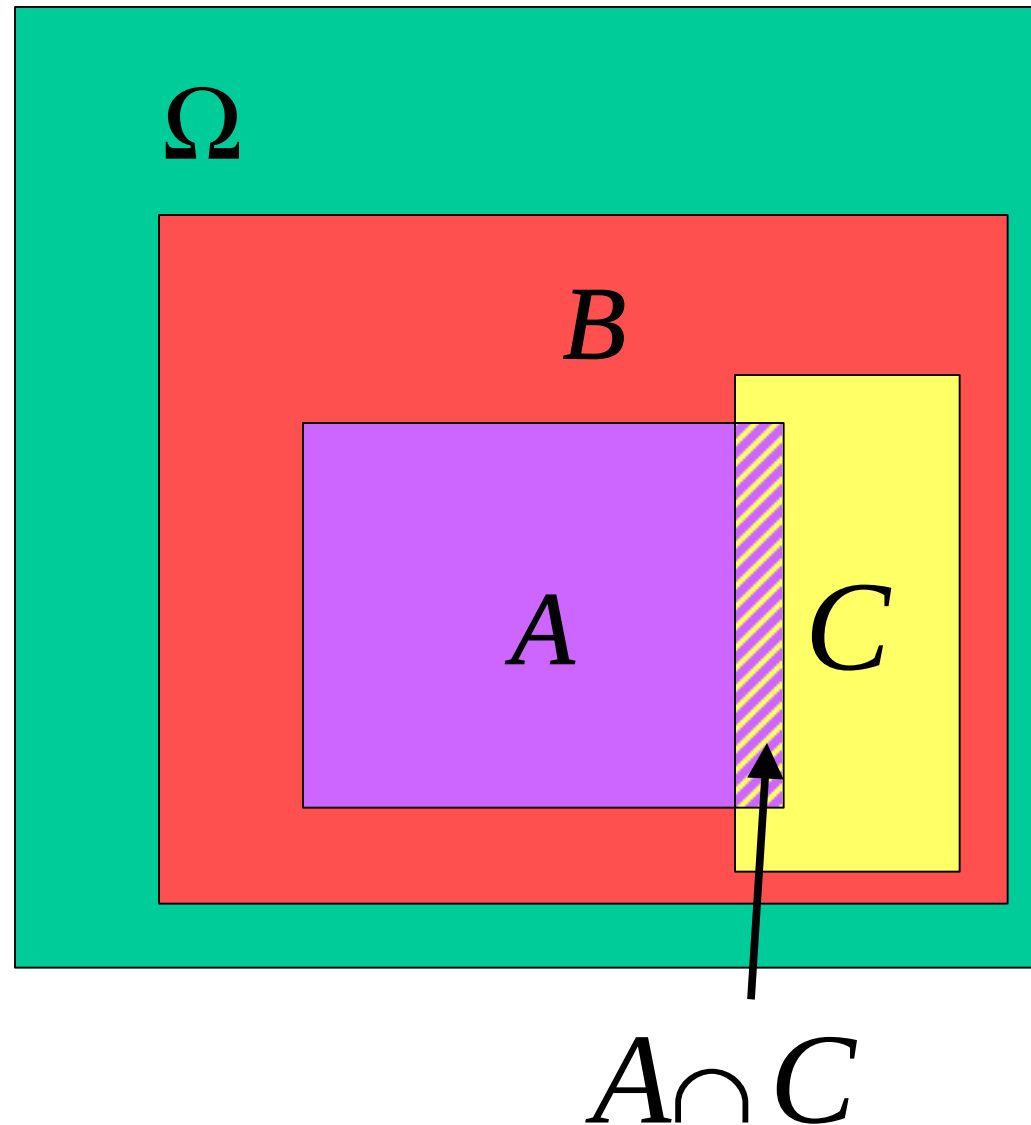
- Consider the event  $B$ : I get at least one even number when I rolled the die 5 times
- All of the elements of  $A$  satisfy  $B$ , so we say that  $A$  is in  $B$ , or included in  $B$

$$A \subset B$$

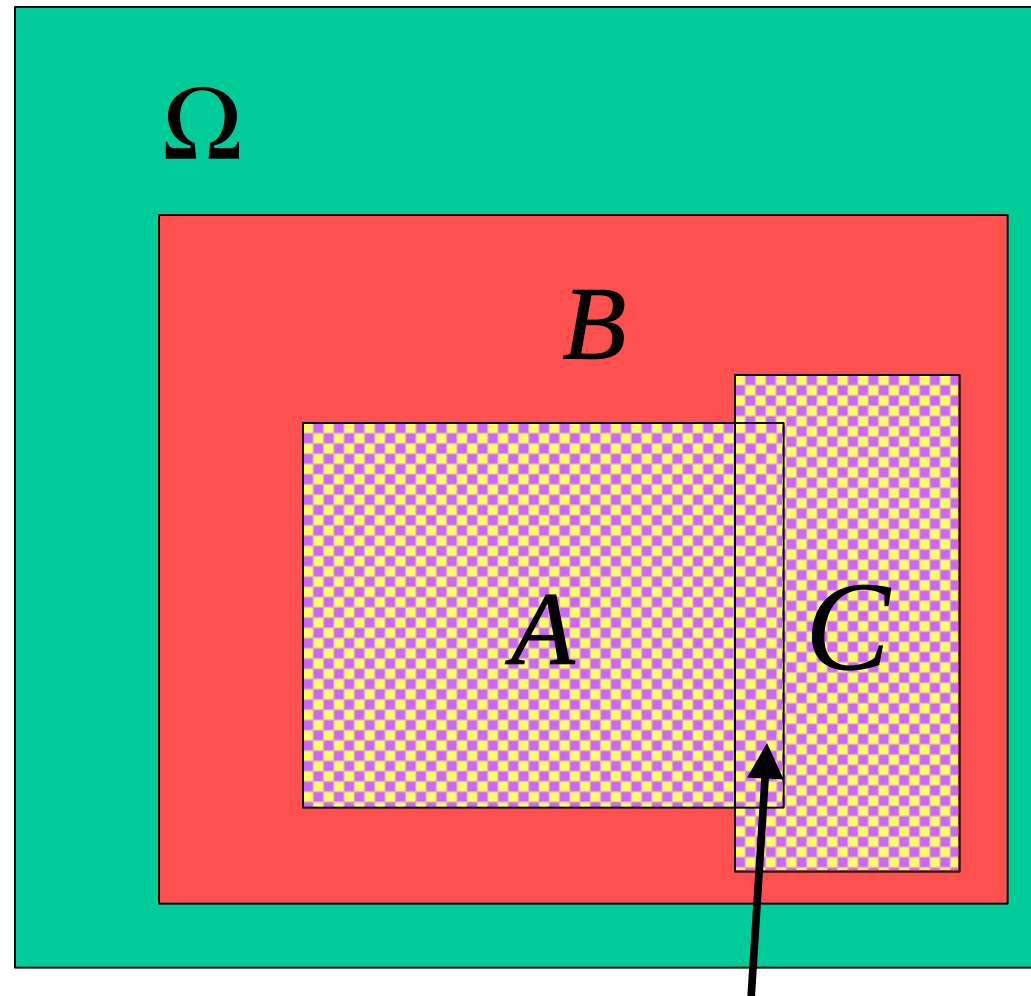
- Consider the event  $C$ : the last roll was a 4
- There is one element in both  $A$  and  $C$ : the set formed by that element alone is the intersection of  $A$  and  $C$

$$A \cap C = \{(1,2,4,5,4)\}$$

# Set Theoretic Notation



# Set Theoretic Notation



Union:  $A \cup C$

# Self-Study

- What is the cardinality of  $C$ ?

$$A \cup C \text{ ?}$$

- Hint: don't double-count the intersection!

# Overview

- Last time:
  - Intro to probability
  - Sample space
  - Events
  - Relating events to sample space: counting
- This time:
  - more counting ... Making our way towards probability

# Think - Pair Share

- What is the cardinality of  $B$ ? Remember, it's I get at least one even number when I rolled the die 5 times

# Think - Pair Share

- What is the cardinality of  $B$ ?
- Sometimes, if we have a very big set, it is easier to look at what is *not* in the set
- To *not* be in  $B$ , you have to get an odd outcome for every roll : there's 3 ways of doing this at every roll
- So  $|B^c| = 3^5$
- So  $|B| = 6^5 - |B^c| = 6^5 - 3^5$

# Counting Elementary Events

- We still consider the same random variable: **what was the outcome when I rolled the die 5 times**
- How many ways are there of selecting 3 elementary events out of  $\Omega$ ?
- First, select the first element: there's  $6^5$  ways
- Next, select the second element: there's  $6^5 - 1$  ways
- Finally, select the last element: there's  $6^5 - 2$  ways
- We selected the items in order - but we are putting them unordered in a bag, there are 6 ways of ordering them, so we have  $(6^5)(6^5 - 1)(6^5 - 2)/6$

# Counting Elementary Events

In general, if we are selecting  
k elements out of a set of n

**n choose k**

$$\binom{n}{k} = n \times (n-1) \times \dots \times (n-k+1) / k!$$

$$= \frac{n!}{k!(n-k)!}$$

Recall

$$n! = n \times (n-1) \times \dots \times 1$$

# Application: the Binomial Theorem

$$(x+y)^n = \underbrace{(x+y) \times \dots \times (x+y)}_{n \text{ times}}$$

- We consider all the possible combinations of products of  $n$  terms
- There is one way of making  $x^n$
- There are ? ways of making  $y x^{n-1}$

# Application: the Binomial Theorem

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- We consider all the possible combinations of products of  $n$  terms
- There is one way of making  $x^n$
- There are  $n$  ways of making  $y x^{n-1}$
- There are ? ways of making  $y^2 x^{n-2}$

# Application: the Binomial Theorem

$$(x+y)^n = \underbrace{(x+y) \times \dots \times (x+y)}_{n \text{ times}}$$

Note: read pages 6-10  
of your  
book along with this

- We consider all the possible combinations of products of  $n$  terms
- There is one way of making  $x^n$
- There are  $n$  ways of making  $y x^{n-1}$
- There are  $n(n-1)/2$  ways of making  $y^2 x^{n-2}$
- In general, there are  $\binom{n}{k}$  ways of making  $y^k x^{n-k}$

# Counting- more advanced topics

- Suppose that we divide  $n$  elements into 3 sets of size  $n_1, n_2, n_3$
- For the first set, there are  $\binom{n}{n_1}$  ways of choosing elements
- For the second set, there are  $\binom{n-n_1}{n_2}$
- For the third, there are  $\binom{n-n_1-n_2}{n_3}$

Note: look in your book  
pages 10-15.

## Counting- more advanced topics

- The total number of ways is the product

$$\binom{n-n_1-n_2}{n_3} \times \binom{n-n_1}{n_2} \times \binom{n}{n_1} = \binom{n}{n_1, n_2, n_3}$$

- This is called the **multinomial coefficient**
- When we write binomial coefficient, we just leave out the second term on the bottom

# Self-Study

- Verify that  $\binom{n}{k} = \binom{n}{n-k}$
- Problem 11 on page 17

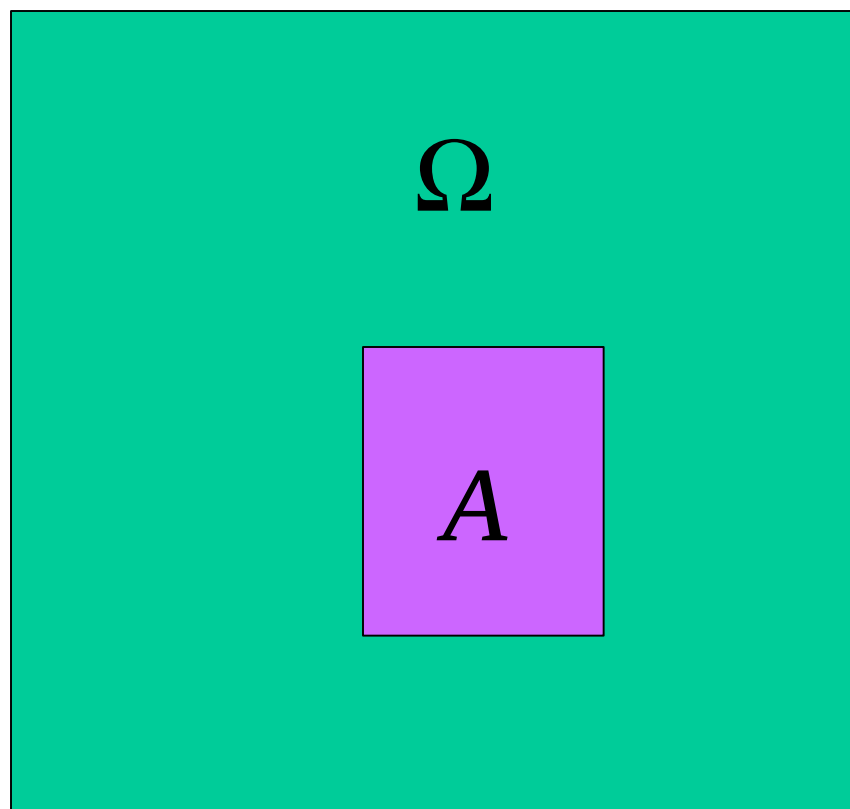
OK - we've been counting things-  
but isn't this a course in **probability**?

# Outline

- Probability axioms
- Venn diagrams and Karnaugh maps

Recall last time ...

Random variable: what was the outcome  
when I rolled the die 5 times



event A:

I had a 1, 2, 4, 5  
and then an  
even number

$$|\Omega| = 6^5$$

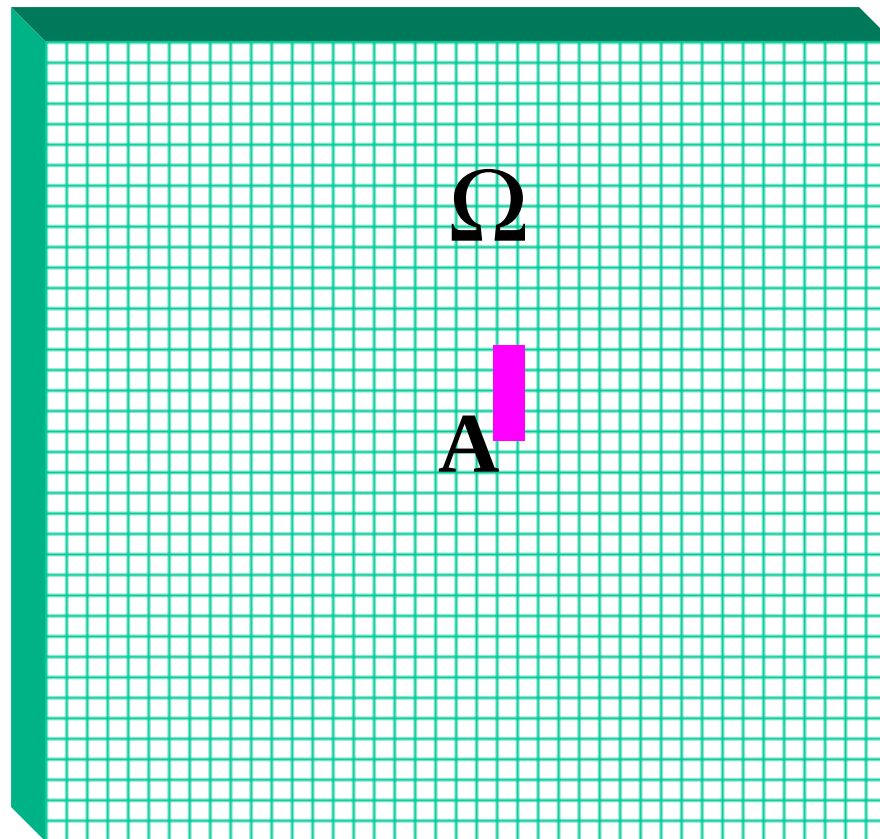
$$|A| = 3$$

$$|A^c| = 6^5 - 3$$

# Let's make it into a chocolate bar

- Each elementary event is a square in the chocolate bar
- At first, we'll consider that the squares all weigh the same
- An event is a piece of chocolate bar (a discrete number of squares)
- If I tell you that your chocolate bar weighs 1 pound, how much does event A weigh? This is the **probability** of event  $A = 3/6^5$

Random variable: what was the outcome  
when I rolled the die 5 times



event A:

I had a 1, 2, 4, 5  
and then an  
even number

$$|\Omega| = 6^5$$

$$|A| = 3$$

$$|A^c| = 6^5 - 3$$

# Axioms (*and their chocolate interpretation*)

- For any event  $E$   $0 \leq P(E) \leq 1$
- *Any piece of the chocolate bar must have a weight which is not negative and no more than a pound*
- $P(\Omega) = 1$
- *The weight of the whole bar is 1 pound*

## Axioms (*and their chocolate interpretation*) contd.

- For any countable sequence  $E_1, E_2, \dots$  of mutually exclusive events, i.e. for all  $i \neq j$   $E_i \cap E_j = \emptyset$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

- *If I take pieces of chocolate which don't share any squares, the weight of all those pieces of chocolate put together is the sum of their individual weights*

# Probability: a more theoretical look

- Probability space  $(\Omega, F, P)$
- $F$  is a  $\sigma$ -field of the events of  $\Omega$ , it includes  $\Omega$ , is closed under complementation and under countable unions and intersections
- Probability measure is a function that assigns real numbers in  $[0, 1]$  to the events in  $F$  subject to the axioms

# Consequences of the axioms

*(and their chocolate interpretation)*

- $P(E^C) = 1 - P(E)$
- *The weight of whatever's left after I take out a piece of chocolate is 1 pound minus the weight of the piece I took out*
- $A \subset B \Rightarrow P(A) \leq P(B)$
- *a piece of chocolate that's part of another piece cannot weigh more than the piece it's part of*

# Reality Check

- What is  $P(\emptyset)$ ?

# Reality Check

- What is  $P(\emptyset)$ ?

**It's 0!**

$$\Omega^C = \emptyset$$

$$\text{so } 1 = P(\Omega) = 1 - P(\emptyset)$$

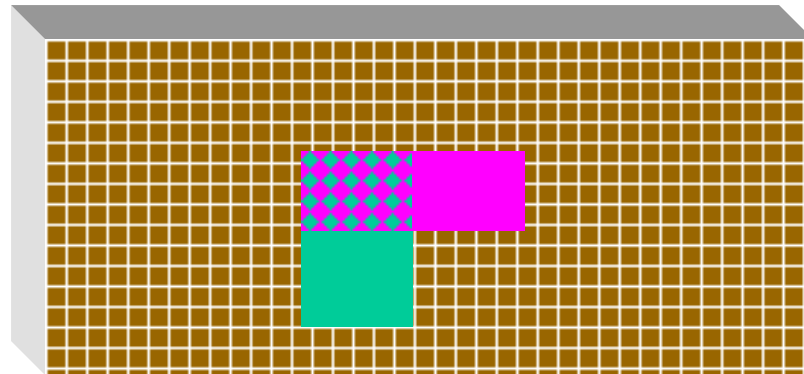
# Consequences of the axioms

*(and their chocolate interpretation)*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- The weight of two pieces of chocolate put together is the sum of the weights minus the weight of whatever squares they have in common (so we don't double count those squares in common)*

Note: this is a special case of 4.4



# DeMorgan's Laws

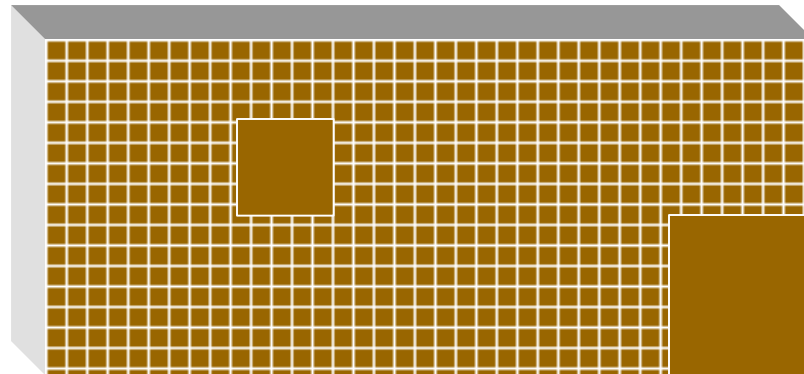
## *(and their chocolate interpretation)*

- $(A \cup B)^C = A^C \cap B^C$
- *The weight of whatever is left after we take out to pieces of chocolate is the weight of whatever was in neither piece of chocolate*
- $(A \cap B)^C = A^C \cup B^C$
- **Self-study:** *write the chocolate interpretation*

Note: after you are comfortable with chocolate interpretations, go through pages 30 - 36 carefully

# Equally likely versus not equally likely

- Recall that we originally assumed that all our **chocolate squares** have the same weight- that is the same as saying that the elementary events all have the same probability
- We could also have assumed that they do not have the same weight



# Self-Study

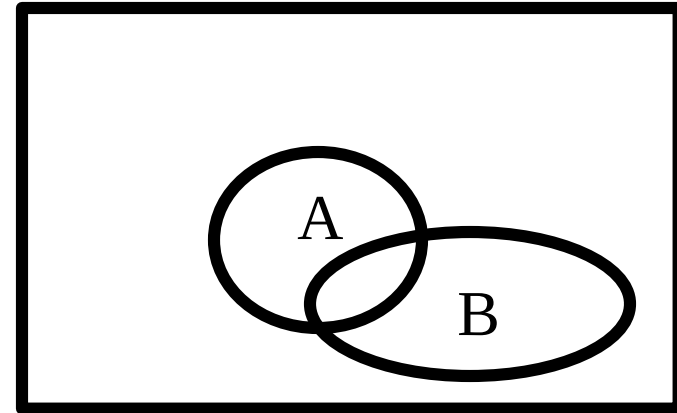
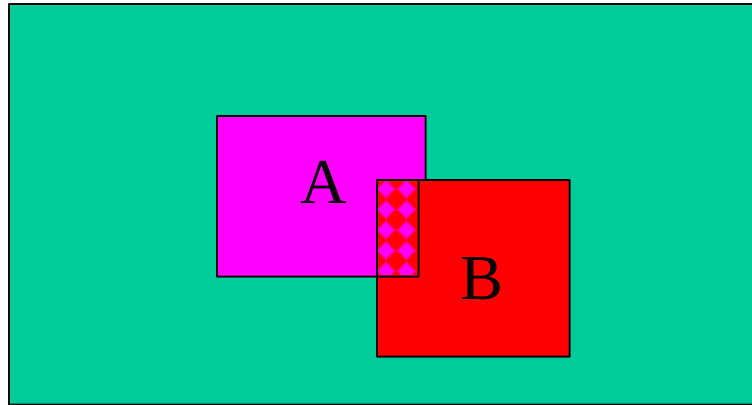
- Go through the axioms and consequences of the axioms - check that nothing depends on actually having equiprobable elementary events (*same size chocolate squares*)
- Read Section 2.5 after you understand the axioms well

# Outline

- Probability axioms
- Venn diagrams and Karnaugh maps

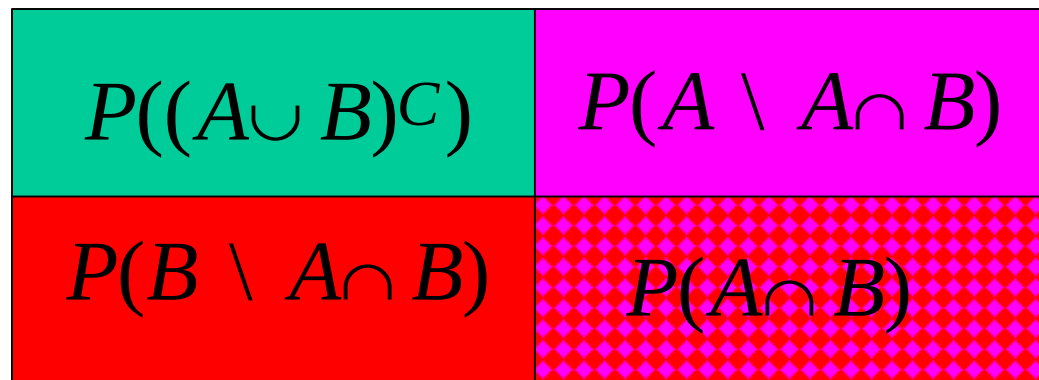
# Venn diagrams and Karnaugh maps

Venn



Venn

Karnaugh



} B

} A

# Putting it all together

- Example: I'm trying to debug a program, A. I am having memory leaks. A calls B and C. There is a register of my memory (register k) which is affected if *both* B and C have memory leaks (if only B or C overflow, the register is unaffected). My experiment is running A and checking the registers before and after. I check register k and it was unaffected
- Event A: A overflows. Event B: B overflows. Event C: C overflows.
- The outcome of the experiment was: if A then B and C did not occur together

# Putting it all together

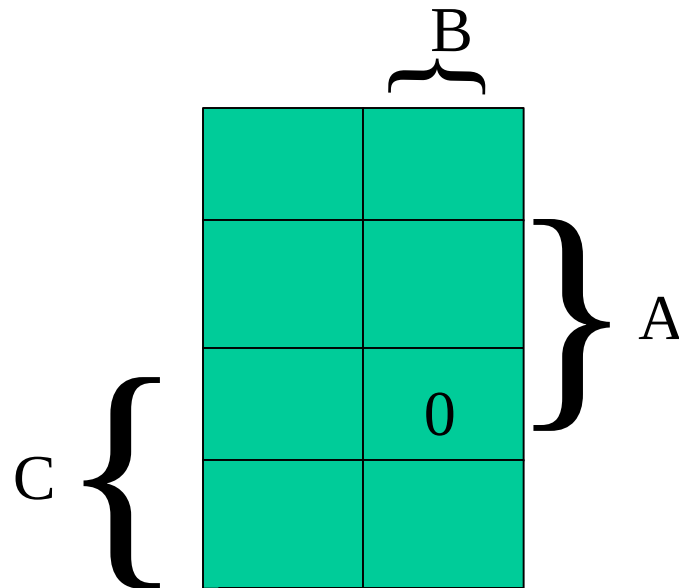
- Random variable: the state of all the registers before and after running A.
- Sample space  $\Omega$ : all the possible states of all the registers before and after running A.
- $|\Omega| = \left( (2^{16})^n \right)^2$  for  $n$  registers of 16 bits each (unless certain states are impossible)
- Elementary event: pair of  $n$ -tuples of 16-bit numbers.

# **Think - Pair Share**

- Give the Karnaugh map

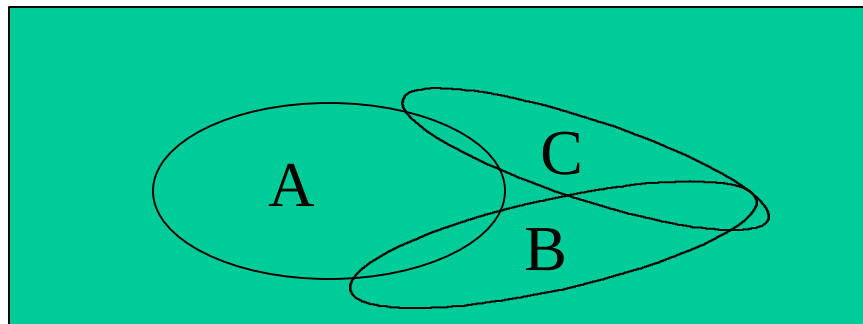
# Think - Pair Share

- Give the Karnaugh map



# Putting it all together

- Event: if A then B and C did not occur together
- Set-theoretic notation:  $(A \cap (B \cap C)^C) \cup A^C$
- Chocolate interpretation: *a square of chocolate in piece A is not in both pieces B and C*
- Venn diagram:



# Does an event with probability 0 never happen?

- In practice, we do not have the accuracy to distinguish between an event which has probability 0 and probability  $10^{-40}$
- If think of a “relative frequency” interpretation (later in the course, we’ll give some backbone to the relative frequency interpretation), then the two are basically the same
- Later we’ll look at continuous outcomes and re-visit the concept of events with probability 0