Assigned: Wednesday, October 6, 1999
Due: Wednesday, October 13, 1999
Reading: Ross, Chapter 5
Noncredit Exercises: You are urged to try all the exercises suggested in Problem Sets 1–5.

Reminder: Hour Exam I is on Monday October 11, 7:00 pm – 8:00 pm in 213 Gregory Hall. More details on the class web page `http://www.ece.uiuc.edu/~ece313/fall99MWF`

Problems:

1. ["One man's meat is another man's Poisson"]

Let $\mathbf{X}$ denote a Poisson random variable with parameter λ.

(a) Show that $P\{\mathbf{X} \text{ is even}\} = \exp(-\lambda)\cosh(\lambda)$. Don't forget that 0 is an even integer!

(b) In Problem 4 of Problem Set #3, you proved (I hope!) that the probability that a binomial random variable with parameters (n,p) has even value is $[1+(1-2p)^n]/2$. For large n and small p, show that $[1+(1-2p)^n]/2 \approx \exp(-np)\cosh(np)$, which is consistent with part (a).

2. ["Take me out to the ballgame"] A baseball pitcher's repertoire is limited to fastballs (event F), curveballs (event C), or sliders (event S). It is known that P(C) = 2P(F), and that the event H that the batter hits the ball has probabilities $P(H|F) = 2/5$, $P(H|C) = 1/4$, and $P(H|S) = 1/6$.

(a) If P(H) = 1/4, what is P(C)?

(b) A fan sitting in the bleachers sees the batter getting a hit, i.e. the event H. He knows the values of $P(H|F)$, $P(H|C)$, and $P(H|S)$, but is sitting too far away to tell whether the pitch was a fastball, a curveball or a slider. What is his maximum-likelihood decision as to what kind of pitch it was?

(c) After cheering the hit, the fan finds that P(F), P(C), and P(S) are listed in the program guide. What is his Bayesian (that is, maximum *a posteriori* probability or minimum-error-probability) decision as to the kind of pitch it was?

3. ["I'm leaving on a prop plane"] Consider again Problem #2 of Problem Set #4. Suppose that 15 of the 105 passengers who hold reservations are arriving in Chicago on a connecting flight. If the connecting flight is on time, all 15 show up for the flight to Champaign (nobody stops off at a bar and misses the flight!); else, obviously none of the 15 shows up. Let $\mathbf{Y}$ denote the number of nonconnecting passengers who actually show up for the flight. Let H_0 denote the hypothesis that the connecting flight is late, and H_1 the hypothesis that the connecting flight is on time. It is reasonable to assume that the pmf of $\mathbf{Y}$ is the same regardless of which hypothesis is true, and hence we model $\mathbf{Y}$ as a binomial random variable with parameters (90, 0.9). On the other hand, $\mathbf{X}$, the *total* number of passengers showing up for the flight, equals $\mathbf{Y}$ if H_0 is true, while if H_1 is true, then $\mathbf{X} = 15 + \mathbf{Y}$, and thus the pmf of $\mathbf{X}$ *does* depend on which hypothesis is true.

(a) Suppose that the gate agent observes that $\mathbf{X} = 86$. What is $P\{\mathbf{X} = 86\}$ when H_0 is the true hypothesis? What is $P\{\mathbf{X} = 86\}$ when H_1 is the true hypothesis? What is the value of the likelihood ratio when $\mathbf{X} = 86$, and what is the agent's maximum-likelihood decision as to whether the connecting flight is late?

(b) Repeat part (a) for the case when the gate agent observes that $\mathbf{X} = 96$.

(c) The gate agent knows that $\pi_0 = P\{H_0 \text{ is the true hypothesis}\} = 1/3$. For each of the two observations considered in parts (a) and (b), what is the agent's MAP (or Bayesian or minimum-probability-of-error} decision as to whether the connecting flight is late?

(d) What is the probability that all passengers who show up get a seat? Given that all passengers who showed up got a seat, find the (conditional) probability that the connecting flight was late.

4. [“It a’in’t about bipartisan politics; it’s about …”] The Senate of a certain country has 100 members consisting of 43 Conservative Republicans, 21 Conservative Democrats, 12 Liberal Republicans, and 24 Liberal Democrats. Before each vote, the groups caucus separately. Each group decides *independently* of the other groups whether to support or oppose the motion. *All* members of the group then vote in accordance with the caucus decision.

For those who think that this is the way politics works, I have this beautiful skyscraper on Wacker Drive in Chicago that I am willing to sell to you at a bargain price…

(a) Let A, B, C, and D respectively denote the events that the four groups vote for a tax bill that will cut taxes by $792 billion. Suppose that the probabilities of these independent events are P(A) = 0.9, P(B) = 0.6 P(C) = 0.5 and P(D) = 0.2. What is the probability that the bill passes?

(b) Let **X** denote the number of votes in favor of the bill. Then, **X** is a discrete random variable. Explain why **X** takes on 16 different values in the range [0,100] and find the pmf of **X**. Compute P{**X** > 50} from the pmf. Is it the same as the answer obtained in part (a)? (Using a spreadsheet/MATLAB/Mathematica will help considerably in doing this part)

(c) The President vetoes the bill. Let E, F, G, and H respectively denote the independent events that the four groups support the motion to override the veto. If these events have probabilities P(E) = 0.99, P(F) = 0.4, P(G) = 0.6, and P(H) = 0.1, what is the probability that the motion to override the veto passes ?

(d) Find the pmf of **X**, the number of votes in favor of overriding the veto.

(e) The Clerk of the Senate, being new on the job, has forgotten whether the vote currently being taken is to pass a bill or to override a veto. The Clerk counts the votes and thus knows the value of **X**. Specify the maximum-likelihood decision rule (as to what kind of vote was just taken) in terms of the observed value of **X**. Thus, your answer should be

“If **X** A then decide that it was a bill, while if **X** B then decide that it was an override”

where A and B are *disjoint* sets of numbers with |A B| = 16.

Political innocents are reminded that a simple majority (51 or more votes) is required to pass a bill, and a two-thirds majority (67 or more votes) to override a veto.

5. [“If God didn’t make little green apples; it don’t rain in Indianapolis all the time…”] We return to the apple-bobbing contest of Problem 1 of Problem Set #2 where Tommy bobs for an apple from the tub containing 8 red apples and 7 green apples.

(a) What is the probability that Tommy gets a red apple?

(b) After Tommy leaves with his apple, the organizers (who dutifully noted the color of Tommy’s apple) *add three apples of the same color* to the tub so that it now contains 17 apples. What is the probability that Linda (who bobs next) gets a red apple? Compare your answer to the answert to part (a).

(c) What is the conditional probability that Tommy got a green apple given that Linda got a red apple?

(d) The organizers once again add three apples of the same color as the one just removed. What is the probability that the next child gets a red apple?

6. [“Tennis, anyone?”] Consider the following simplified model for a game of tennis. On each serve, let p denote the probability that player A wins the point, and q = 1–p the probability that player B wins the point. Assume that the outcome of each serve is independent of all others. Player A wins the game if the score reaches 4–0, 4–1, or 4–2, while B wins the game if the score reaches 2–4, 1–4, or 0–4. Else, the score reaches 3–3 (called deuce) and from this point onwards, the game continues until one player is two points ahead of the other, and thereby wins the game.

(a) Find the probabilities that the score reaches 4–0, 4–1, or 4–2 and the probabilities that the score reaches 2–4, 1–4, or 0–4. I need 6 answers here!

(b) Find P(score reaches deuce).

(c) Find the numerical values of the seven probabilities obtained in parts (a) and (b) for p = 0.2, 0.4, 0.5, 0.6, and 0.8 and verify that the sum equals 1 in each case.

(d) Given that the score is deuce, what is P(A wins the next two points)? (This means A wins the game). What is P(B wins the next two points)? (This means B wins the game). What is the probability that both players win one point each? In this case, the score is tied again, and is also called deuce. Answers in terms of the parameter p, please: no numerical values!

(e) Once the score reaches deuce, there *may* be further deuces until ultimately, either A or B wins both points and thereby wins the game. What is the probability that A ultimately wins the game given that the score is deuce? What is the probability that B ultimately wins the game given that the score is deuce? (Hint: these answers are different from those of part (d)) What is the probability that the game goes on forever with the score continuing to reach deuce after every two points?

(f) Use the results of parts (a)–(e) to express the probability that A wins the game as a function f(p) of p. A little thought shows that B wins with probability f(q) = f(1–p). Now, if p = 0, A wins no points which makes it difficult for him to win any games. Does your function f(p) satisfy f(0) = 0? If not, what does your f(p) give as the probability that A wins a game while losing every point? Similarly, if A wins every point, he is sure to win the game. Does your function f(p) satisfy f(1) = 1? If not, what does your f(p) give as the probability that A loses a game while winning every point?
Other reasonable properties of f(p) are f(0.5) = 0.5, f(p) + f(1–p) =1. Which of these is satisfied by your function f(p)?

(g) Expand f(p) in a Taylor series in the neighborhood of p = 0.5 (only the first two terms are needed) What does this say about the probability of winning a game if $p = 0.5 + \varepsilon$ where ε is very small?

(h) Use your favorite graphing program to sketch f(p) as a function of p for $0 \leq p \leq 1$. Determine the minimum value of p for which $f(p) \geq 2/3$.

7. The dice game of craps (see Ross, p. 58) begins with the player (called the shooter) rolling two fair dice. If the result is a 2, or 3, or 12, the shooter loses, while if the result is a 7 or 11, the shooter wins.

(a) What is the probability that the shooter loses on the first roll? What is the probability that the shooter wins on the first roll?

(b) If the sum of the dice on the **first roll** is any of 4, 5, 6, 8, 9, 10, that number is called the **shooter's point.** For **each** number i in the set {4, 5, 6, 8, 9, 10}, find the probability that the shooter's point is i. I need six answers here, folks!

(c) Suppose that the shooter's point is i where i is some number in {4, 5, 6, 8, 9, 10}. The shooter now rolls the two dice again. If the result is a 7, the shooter loses (craps out.)
If the result is i, the shooter wins (this is referred to as making the point). If the result is neither i nor 7, the shooter rolls again. This process continues until the shooter either makes the point or craps out. Given that the shooter's point is i, what is the conditional probability that the shooter makes the point? Naturally, the answer depends on i, so here too, I need six answers.

(d) Use the above results to compute the probability of winning at craps.

(e) Given that the shooter's point is 8, what is the probability that the shooter makes it "the hard way," that is, by rolling two fours? Generally, bets are offered at 10-to-1 odds that the shooter does not make the point 8 the hard way. That is, if you bet $1, you win $10 (plus your $1 back!) if the shooter makes 8 the hard way; and you lose the $1 that you bet if the shooter craps out or makes 8 by rolling 2-6, 3-5, 5-3, or 6-2). In the long run over many such bets, do you expect to make money, or lose money, or come out even?