University of Illinois **Final Exam Solutions** **ECE 313**

 Fall 1999

1.(a) P(A|B) > P(A) is **FALSE**. Conditional probabilities can be smaller than unconditional probabilities.

P(A|B) + P(A|B^c) = 1 is **FALSE**. If A   B, then P(A|B^c) = 0 while P(A|B) = P(A)/P(B) ≠ 1 in all cases.

P(A|B) + P(A^c|B^c) = 1 is **FALSE**. If A ∩ B =   , then P(A|B) = 0 and P(A^c|B^c) ≠ 1 in all cases.

P(A|B) + P(A^c|B) = 1 is **TRUE**. Conditional probabilities are a probability measure.

If P(A) = P(B), then P(A|B) = P(B|A) is **TRUE**. P(A|B) = P(A ∩ B)/P(B) = P(A ∩ B)/P(A) = P(B|A).

If P(A|B) = P(B|A), then P(A) = P(B) is **FALSE**! If A ∩ B =   , then P(A|B) = P(B|A) = 0; P(A) ≠ P(B).

If P(A|B) = P(B|A), then A and B are independent is **FALSE**. Independence requires P(A ∩ B) = P(A)P(B)

P(A ∩ B) ≤ P(A)P(B) with equality if and only if A and B are independent is **FALSE.** Note that the alleged result is equivalent to P(A|B) ≤ P(A) which is just the complement of the first question.

P(B|A)P(A) + P(A|B^c)P(B^c) = P(A) is **TRUE**. The first term equals P(A|B)P(B).

P(A|B) = P(B|A)P(B)/P(A) is **FALSE**. If If A ∩ B ≠   , it holds only if P(A) = P(B).

(b) ‹ P(A ∩ B) ≤ min{P(A), P(B)}   ` P(A   B) ≥ [P(A) + P(B)]/2

´ P(A ∩ B) ≥ P(A) + P(B) – 1.   ˆ P(A ∩ B) ≤ P(A|B)

Since A ∩ B   A, P(A ∩ B) ≤ P(A). Similarly, P(A ∩ B) ≤ P(B)   P(A ∩ B) ≤ min{P(A), P(B)}.

Since A   A   B, P(A) ≤ P(A   B). Similarly, P(B) ≤ P(A   B); hence,P(A   B) ≥ [P(A) + P(B)]/2.

Since 1 ≥ P(A   B) = P(A) + P(B) – P(A ∩ B), it follows that P(A ∩ B) ≥ P(A) + P(B) – 1.

Since P(B) < 1, it follows that P(A ∩ B) = P(A|B)P(B) ≤ P(A|B). Thus, all four statements are true.

(c) ‹ P{**X** > b} = 1 – $F_{\mathbf{X}}(b)$.   ` If $F_{\mathbf{X}}(a) < F_{\mathbf{X}}(b)$, then a < b.

´ If a < b, then $F_{\mathbf{X}}(a) < F_{\mathbf{X}}(b)$.   ˆ $F_{\mathbf{X}}(u) = 1/2$ for some u, –∞ < u < ∞.

The first two properties hold for all CDFs since $F_{\mathbf{X}}(u)$ is a *right-continuous non-decreasing* function. But, if a < b, it is possible that $F_{\mathbf{X}}(a) = F_{\mathbf{X}}(b)$ because the CDF did not increase between those points (it cannot decrease, of course.) As a counterexample for ˆ , note that for a Bernoulli random variable with parameter p ≠ 1/2, $F_{\mathbf{X}}(u)$ takes on values 0, p, and 1 only. Thus, ´ and ˆ are not properties of all CDFS.

(d) **X** and **Y** have identical finite variance σ^2

‹ $E[\mathbf{X}^2] = E[\mathbf{Y}^2]$ ` var(4**X** – 5**Y**) = var(4**Y** – 5**X**)

´ |cov(**X**, **Y**)| ≤ σ^2 ˆ **X** + **Y** and **X** – **Y** are uncorrelated RVs

$E[\mathbf{X}^2] = \sigma^2 + \{E[\mathbf{X}]\}^2 \neq E[\mathbf{Y}^2] = \sigma^2 + \{E[\mathbf{Y}]\}^2$ unless it so happens that E[**X**] = E[**Y**].

var(4**X** – 5**Y**) = 16•var(**X**) + 25•var(**Y**) –2•4•5•cov(**X**,**Y**) = 41σ^2 – 40•cov(**X**,**Y**).

var(4**Y** – 5**X**) = 16•var(**Y**) + 25•var(**X**) –2•4•5•cov(**Y**,**X**) = 41σ^2 – 40•cov(**X**,**Y**). Hence, equality holds.

$|\mathrm{cov}(\mathbf{X},\mathbf{Y})| = |\rho\sqrt{\mathrm{var}(\mathbf{X})\mathrm{var}(\mathbf{Y})}\,| = |\rho|\bullet\sigma^2 \leq \sigma^2$ since $|\rho| \leq 1$.

cov(**X** + **Y**, **X** – **Y**) = var(**X**) – var(**Y**) – cov(**X**,**Y**) + cov(**Y**,**X**) = $\sigma^2 - \sigma^2$ – cov(**X**,**Y**) + cov(**X**,**Y**) = 0.

Thus, ` , ´ , and ˆ are true statements.

2. 5 arrivals occur in (0,4], 2 in (3,4] and 4 in (3,6]. These events are not independent, but all three occur if and only if there are 3 arrivals in (0,3], 2 arrivals in (3,4], and 2 arrivals in (4,6]. The latter three events *are* independent since they correspond to disjoint time intervals. Hence,

P{**X** = 5, **Z** = 4 | **Y** = 2} = P{**X** = 5, **Z** = 4, **Y** = 2}/P{**Y** = 2}

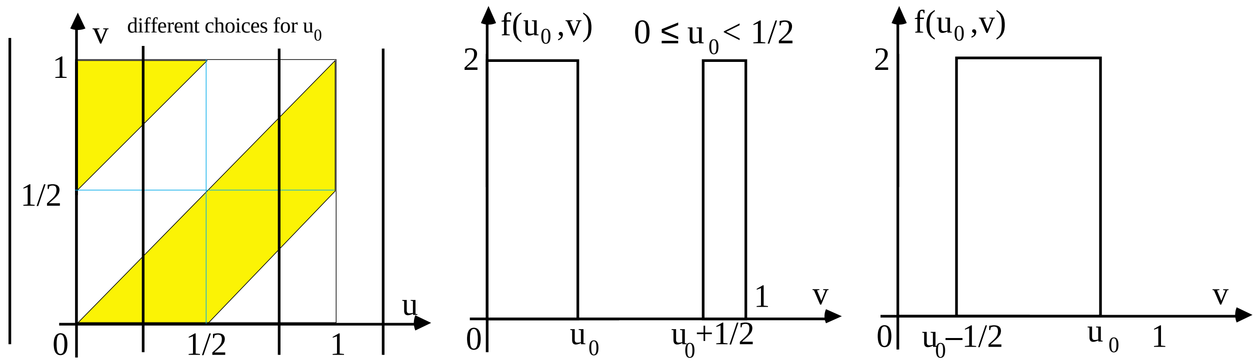
= P{3 arrivals in (0,3], 2 arrivals in (3,4], and 2 arrivals in (4,6]}/P{2 arrivals in (3,4]}

= P{3 arrivals in (0,3], 2 arrivals in (4,6]} = P{3 arrivals in (0,3]}P{2 arrivals in (4,6]}

$$= \frac{\exp(-3\lambda)\bullet(3\lambda)^3}{3!}\,\frac{\exp(-2\lambda)\bullet(2\lambda)^2}{2!} = 9\bullet\exp(-5\lambda)\bullet(\lambda)^5.$$

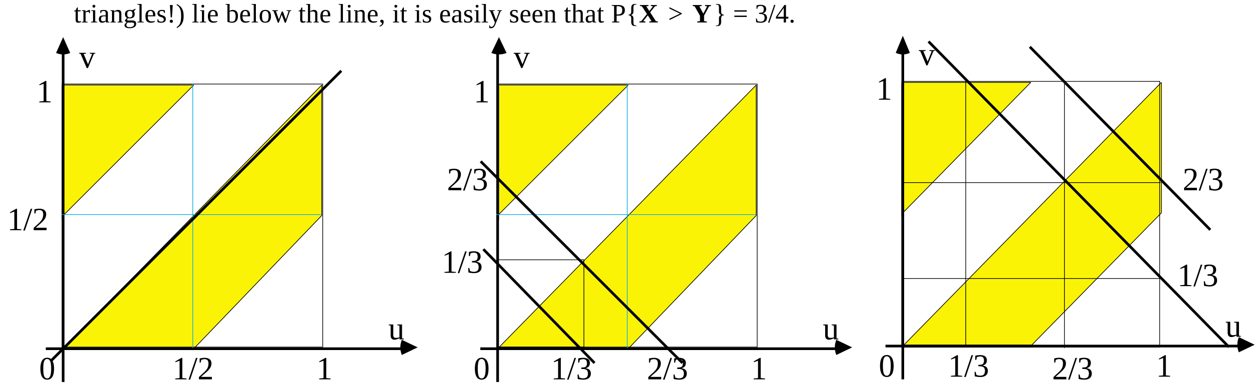
3. This is a Gaussian random variable with mean 0 and variance $(1/2\pi)$!!
Hence, $E[\mathbf{X}^2 + 2\mathbf{X} + 3] = E[\mathbf{X}^2] + 2\bullet E[\mathbf{X}] + 3 = [(1/2\pi) + 0^2] + 2\bullet 0 + 3 = (1/2\pi) + 3$.
$P\{|\mathbf{X}| > \sqrt{2/\pi}\} = P\{|\mathbf{X}| > 2\sigma\} = 2\bullet\Phi(-2) = 2\bullet[1-\Phi(2)] = 2\bullet[1-0.9772] = 0.0456$.

4.(a) $f_{\mathbf{X}}(u_0)$, the value of the marginal pdf at any *fixed number* u_0 is the area of the cross-section of the pdf surface at $u = u_0$, that is, $f_{\mathbf{X}}(u_0)$ equals the area under the curve $f_{\mathbf{X},\mathbf{Y}}(u_0,v)$ regarded as a function of v. As can be observed from the diagrams below, this cross-section is 0 for $u_0 < 0$ or $u_0 > 1$. On the other hand, the cross-sections look as shown below for $0 \le u_0 < 1/2$ and for $1/2 \le u_0 \le 1$.



Since the cross-section areas are easily shown to be 1, it follows that $f_{\mathbf{X}}(u_0) = 1$ for all choices of u_0 in the range [0, 1] and thus $f_{\mathbf{X}}(u) = \begin{cases} 1, & 0 \le u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$ This is obviously a valid pdf: **X** is uniformly distributed on the interval [0,1]. A similar calculation shows that **Y** is also uniformly distributed on [0,1].

(b) $P\{\mathbf{X} > \mathbf{Y}\}$ is the probability that the random point $(\mathbf{X},\mathbf{Y})$ lies below the heavy line in the left-hand figure below. Since the joint pdf is uniform on the shaded region and three out of four dentists (oops, I mean triangles!) lie below the line, it is easily seen that $P\{\mathbf{X} > \mathbf{Y}\} = 3/4$.



(c) No, **X** and **Y** are not independent which can be proved via the "eyeball test" viz. the pdf is nonzero on a nonrectangular region, or from noting that, on the unit square, $f_{\mathbf{X}}(u)f_{\mathbf{Y}}(v) = 1 \ne f_{\mathbf{X},\mathbf{Y}}(u,v)$.

(d) $P\{\mathbf{X} + \mathbf{Y} < 1/3\}$ is the probability that the random point $(\mathbf{X},\mathbf{Y})$ lies below the heavy line (with intercepts 1/3 and 1/3 on the axes) in the middle figure below. Since the area of the shaded region is one-fourth the area of the square of side (1/3), we get that $P\{\mathbf{X} + \mathbf{Y} < 1/3\}$ = integral of joint pdf over region below line
= (value of joint pdf on shaded region)•area = $2\bullet(1/4)\bullet(1/3)^2 = (1/2)\bullet(1/3)^2 = 1/18$.
Similarly, $P\{\mathbf{X} + \mathbf{Y} < 2/3\}$ is the probability that the random point $(\mathbf{X},\mathbf{Y})$ lies below the heavy line (with intercepts 2/3 and 2/3 on the axes) in the middle figure above. Since the "little extra upper piece" fits neatly into the "little empty space below", we see that $P\{\mathbf{X} + \mathbf{Y} < 2/3\} = 4\bullet P\{\mathbf{X} + \mathbf{Y} < 1/3\} = 2/9 = (1/2)\bullet(2/3)^2$.
In fact, for any real number α, $0 \le \alpha \le 1$, $P\{\mathbf{X} + \mathbf{Y} < \alpha\} = (1/2)\bullet(\alpha)^2$. Quick check: the formula gives a probability of 0 for $\alpha = 0$ and a probability of 1/2 for $\alpha = 1$ which checks with the obvious results.
$P\{\mathbf{X} + \mathbf{Y} > 4/3\}$ is the probability that the random point $(\mathbf{X},\mathbf{Y})$ lies above the heavy line (with (unmarked) intercepts 4/3 and 4/3 on the axes) in the right-hand figure above. A little thought shows that this probability is in fact equal to $P\{\mathbf{X} + \mathbf{Y} < 2/3\} = 2/9$ and similarly $P\{\mathbf{X} + \mathbf{Y} > 5/3\} = P\{\mathbf{X} + \mathbf{Y} < 1/3\} = 1/18$. More generally, for any real number α, $1 \le \alpha \le 2$, $P\{\mathbf{X} + \mathbf{Y} > \alpha\} = P\{\mathbf{X} + \mathbf{Y} < 2-\alpha\} = (1/2)\bullet(2-\alpha)^2$. Quick

check: the formula gives a probability of 1/2 for $\alpha = 1$ and a probability of 0 for $\alpha = 2$ which checks with the obvious results.

(e) From ther results of part (d) it follows that if we set $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$, then $F_{\mathbf{Z}}(\alpha) = P\{\mathbf{Z} \le \alpha\} = P\{\mathbf{X} + \mathbf{Y} \le \alpha\}$ is given by

$$F_{\mathbf{Z}}(\alpha) = \begin{cases} 0, & \alpha < 0, \\ (1/2)\cdot(\alpha)^2, & 0 \le \alpha < 1, \\ 1 - (1/2)\cdot(2-\alpha)^2, & 1 \le \alpha < 2, \\ 1, & \alpha \ge 2, \end{cases} \qquad \text{while } f_{\mathbf{Z}}(\alpha) = \begin{cases} \alpha, & 0 \le \alpha < 1, \\ 2-\alpha, & 1 \le \alpha < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Note that for non-independent random variables, the pdf of the sum is, *in general*, ***not*** the convolution of the marginal pdfs. However, in this instance, even though **X** and **Y** are not independent, $f_{\mathbf{Z}}$ serendipitiously does happen to equal $f_{\mathbf{X}} * f_{\mathbf{Y}}$.

5.(a) $E[\mathbf{W}] = 200$, $\text{var}(\mathbf{W}) = \frac{30^2}{12} = 75$, $E[\mathbf{H}] = 70$, $\text{var}(\mathbf{H}) = \frac{6^2}{12} = 3$.

Hence, $E[\mathbf{WH}] = \text{cov}(\mathbf{W},\mathbf{H}) + E[\mathbf{W}]E[\mathbf{H}] = \rho\cdot\sqrt{\text{var}(\mathbf{W})\text{var}(\mathbf{H})} + 14{,}000 = \frac{1}{3}\cdot\frac{30\cdot 6}{12} + 14{,}000 = 14{,}005$.

(b) $E[\mathbf{M}] = E[\mathbf{W}-\mathbf{H}] = E[\mathbf{W}] - E[\mathbf{H}] = 200 - 70 = 130$.
$\text{var}(\mathbf{M}) = \text{var}(\mathbf{W}-\mathbf{H}) = \text{var}(\mathbf{W}) + \text{var}(\mathbf{H}) - 2\cdot\text{cov}(\mathbf{W},\mathbf{H}) = 75 + 3 - 10 = 68$.

(c) The pdf of $\mathbf{S} = \mathbf{W} + \mathbf{H}$ cannot be found from the given information because the joint pdf is not given. The marginal pdfs and the correlation coefficient are not enough to determine the joint pdf.

6.(a) **X** is a Bernoulli random variable with parameter $p = 0.8$. Its mean is $p = 0.8$ and its variance is $p(1-p) = 0.16$ These quantities can be readily computed even if they are not on your sheets of notes.

(b) **Y** is a random variable with mean $10000\cdot 0.8 = 8{,}000$ and variance $10000\cdot 0.16 = 1600 = 40^2$. The Chebyshev bound gives $P\{|\mathbf{Y} - 8{,}000| \ge 80\} = P\{|\mathbf{Y}-\mu| \ge 2\sigma\} \le 1/2^2 = 1/4$, and therefore $P\{|\mathbf{Y} - 8{,}000| < 80\} = P\{7920 < \mathbf{Y} < 8080\} \ge 3/4 = 0.75$

(c) The Central Limit Theorem allows us to treat **Y** as approximately a Gaussian random variable with mean 8,000 and variance 1600. Hence, $P\{7920 < \mathbf{Y} < 8080\} \approx \Phi(2) - \Phi(-2) = \Phi(2) - [1 - \Phi(2)] = 0.9544$. Note that the range includes the central lobe and thus this is an appropriate use of the Central Limit Theorem.

7. $P\{\mathbf{Y} > 3\mathbf{X}\} = P\{3\mathbf{X} - \mathbf{Y} < 0\}$. But, $3\mathbf{X} - \mathbf{Y}$ is a linear combination of jointly Gaussian random variables and is thus also a *Gaussian* random variable. Its mean is $E[3\mathbf{X} - \mathbf{Y}] = 3E[\mathbf{X}] - E[\mathbf{Y}] = 3\cdot 0 - 7 = -7$ and its variance is $\text{var}(3\mathbf{X} - \mathbf{Y}) = 9\cdot\text{var}(\mathbf{X}) + \text{var}(\mathbf{Y}) - 2\cdot 3\cdot\text{cov}(\mathbf{X},\mathbf{Y}) = 9\cdot 4 + 16 - 6\cdot(1/16)\cdot 2\cdot 4 = 49 = 7^2$. Hence, $P\{3\mathbf{X} - \mathbf{Y} < 0\} = \Phi((0-(-7)/7) = \Phi(1) = 0.8413$.