Two different versions of the exam were used: numerical values of the answers depend on the version.

Problem 1.

A. Only S2 is true. **B.** S3 is always true; S4 is always false. **C.** S5 may be true but S6 is never true.

Problem 2.

(a) The number of arrivals in the interval (0, 4] is a Poisson random variable **X** with parameter 4λ. Hence, the mean number of arrivals is $E[\mathbf{X}] = 4\lambda$.

(b) Consider 2 disjoint intervals (0,2] and (2,6]. The events in these intervals are independent. Hence, P{three arrivals in (0, 3] AND no arrivals in (2,6]} = P{3 arrivals in interval (0,2] AND none in (2,6]}
$= \left[\exp(-2\lambda)\frac{(2\lambda)^3}{3!}\right] \times \exp(-4\lambda) = \exp(-6\lambda)\frac{(2\lambda)^3}{3!}$

(c) The number of arrivals in (0, 6] is a Poisson random variable **X** with parameter 6λ. The event $\{\mathbf{X} = 5\}$ has probability $\exp(-6\lambda)\frac{(6\lambda)^5}{5!}$, and this is maximized if $\lambda = 5/6$ as shown on homework previously.

(d) $\lambda = \ln a$. P{that at least one arrival in (0, t]} = 1 – P{no arrivals} = $1 - \exp(-\lambda t) = 1 - a^t$.

Problem 3.

A False. If $F(u) = F(-u)$ for all u, then $F(\infty) = F(-\infty)$. But $F(\infty) = 1$ and $F(-\infty) = 0$.

B True. The symmetry of the pdf about 0 implies that for any $t > 0$, $h(t) = \int_0^t f(u)du = \int_{-t}^0 f(u)du$ (Draw a sketch and puzzle it out!).

Hence, for $t > 0$, $F(t) = \int_{-\infty}^t f(u)du = \int_{-\infty}^0 f(u)du + \int_0^t f(u)du = 1/2 + \int_0^t f(u)du = 1/2 + h(t)$

while $F(-t) = \int_{-\infty}^{-t} f(u)du = \int_{-\infty}^0 f(u)du - \int_{-t}^0 f(u)du = 1/2 - \int_{-t}^0 f(u)du = 1/2 - h(t)$.

Define $g(t) = F(t) - 1/2$ for all t. Then, for $t > 0$, $g(t) = [F(t) - 1/2] = h(t) = -[F(-t) - 1/2] = -g(-t)$.

C True. $P\{|\mathbf{X}| > a\} = \int_{-\infty}^{-a} f(u)du + \int_a^{\infty} f(u)du = 2\int_{-\infty}^{-a} f(u)du = 2F(-a)$.

Problem 4.

A If one remembers the fundamental result that a linear function of a Gaussian random variable is also a Gaussian variable, then it follows from the further remembrance of $E[\mathbf{Y}] = E[2\mathbf{X} + a] = 2E[\mathbf{X}] + a$ that $E[\mathbf{Y}] = 2 + a$ and that $\mathrm{var}(\mathbf{Y}) = \mathrm{var}(2\mathbf{X} + a) = 2^2\mathrm{var}(\mathbf{X}) = 16 = 4^2$, so that $\mathbf{Y} \sim N(2 + a, 4^2)$, or more formally, $f_{\mathbf{Y}}(v) = (1/4\sqrt{2\pi})\bullet\exp(-(v-2-a)^2/32)$. However, if these results are not remembered, it is still easy to write that $F_{\mathbf{Y}}(v) = P\{\mathbf{Y} \le v\} = P\{2\mathbf{X}+a \le v\} = P\{\mathbf{X} \le (v-a)/2\} = \Phi\left(\frac{(v-a)/2-1}{2}\right) = \Phi\left(\frac{v-a-2}{4}\right)$ since **X** is Gaussian with mean 1 and variance 4. This shows **Y** is also a Gaussian random variable with mean a + 2 and variance 16, i.e. $\mathbf{Y} \sim N(2 + a, 4^2)$. Differentiators should note that the derivative of $\Phi(x)$ is the unit Gaussian pdf $(1/\sqrt{2\pi})\bullet\exp(-x^2/2)$ which, with the "chain rule," gives the pdf $f_{\mathbf{Y}}(v)$ explicitly.

B Since $E[\mathbf{Y}] = E[2\mathbf{X} + a] = 2E[\mathbf{X}] + a = 2 + a$, and $\mathrm{var}(\mathbf{Y}) = \mathrm{var}(2\mathbf{X} + a) = 2^2\mathrm{var}(\mathbf{X}) = 16$, it follows that $E[\mathbf{Y}^2] = \mathrm{var}(\mathbf{Y}) + \{E[\mathbf{Y}]\}^2 = 16 + (2 + a)^2$. This calculation would be valid even if **X** were not Gaussian. It is also possible to obtain the result directly as follows: $E[\mathbf{Y}^2] = E[(2\mathbf{X} + a)^2] = 4E[\mathbf{X}^2] + 4a\bullet E[\mathbf{X}] + a^2 = 4(\mathrm{var}(\mathbf{X}) + \{E[\mathbf{X}]\}^2) + 4a\bullet E[\mathbf{X}] + a^2 = 20 + 4a + a^2 = 16 + (2 + a)^2$.

Problem 5.

Let p be the probability of a Pikachu card and $q = 1-p$ that of a Charmander card.

A $P\{\mathbf{X} = 1\}$ = P{PCC} + P{CPC} + P{CCP} + P{CCCP} + P{CCCCP} + …
$= 3pq^2 + pq^3 + pq^4 + \ldots = 3pq^2 + pq^3[1 + q + \ldots] = 3pq^2 + pq^3/(1 - q) = q^2(2p + 1)$.
More directly, $\mathbf{X} = 1$ if and only if the first two cards are CC (why?) , or the first three cards are PCC or CPC. Hence, $P\{\mathbf{X} = 1\}$ = P{CC} + P{PCC} + P{CPC} $= q^2 + 2pq^2 = q^2(2p + 1)$ as before.

B $P\{\mathbf{Z} = 0\} = P\{\mathbf{Z} = 1\} = P\{\mathbf{Z} = 2\} = 0$ since at least three boxes of Soggies must be purchased.
Now, $P\{\mathbf{Z} = 3\}$ = P{PCC} + P{CPC} + P{CCP} + P{PPC} + P{PCP} + P{CPP} $= 3pq^2 + 3p^2q$.
More than three boxes are purchased if and only if the first three boxes resulted in CCC or PPP. Thus, for larger values of k, we have a distribution which is akin to geometric: $P\{\mathbf{Z} = k\} = pq^{k-1} + p^{k-1}q$ for $k > 3$.
Check: $P\{\mathbf{Z} > 3\}$ = P{CCC} + P{PPP} $= q^3 + p^3$ according to our analysis. It is easy to compute the sum of $P\{\mathbf{Z} = k\}$ for $k > 3$ to get $\Sigma\, pq^{k-1} + p^{k-1}q = q^3 + p^3$, so we have not lost anybody along the way.
Finally, $P\{\mathbf{Z} = 3\} + P\{\mathbf{Z} > 3\} = \{3pq^2 + 3p^2q\} + q^3 + p^3 = (p + q)^3$, so we have a valid pmf.