University of Illinois | **First Hour Exam** | **ECE 313 Fall 1999**

Two different versions of the exam were used: numerical values of the answers depend on the version.

1. There are n red and n black balls in the urn.

(a) $(R_1 \quad R_2) \cap (B_1 \quad B_2) = (R_1 \cap B_2) \quad (B_1 \cap R_2)$ since $R_1 \cap B_1 = R_2 \cap B_2 = \quad$. Hence,

$P\{(R_1 \quad R_2) \cap (B_1 \quad B_2)\} = P(R_1)P(B_2 | R_1) + P(B_1)P(R_2 | B_1) = \frac{n}{2n} \times \frac{n}{2n-1} + \frac{n}{2n} \times \frac{n}{2n-1} = \frac{n}{2n-1}$.

$P\{B_1 | (R_1 \quad R_2)\} = \frac{P\{B_1 \cap (R_1 \quad R_2)\}}{P\{R_1 \quad R_2\}} = \frac{P\{B_1 \cap R_2\}}{P\{R_1 \quad R_2\}}$. We have already found

$P\{B_1 \cap R_2\} = \frac{n}{2n} \times \frac{n}{2n-1}$. Since $P\{R_1 \quad R_2\} = 1 - P\{B_1 \cap B_2\} = 1 - \frac{n}{2n} \times \frac{n-1}{2n-1} = \frac{n(3n-1)}{2n(2n-1)}$

we get that $P\{B_1 | (R_1 \quad R_2)\} = \frac{n}{3n-1}$.

(b) R_1 and R_2 are neither disjoint nor independent. However,

$P(R_2) = P(R_2 | B_1)P(B_1) + P(R_2 | R_1)P(R_1) = \frac{n}{2n-1} \times \frac{n}{2n} + \frac{n-1}{2n-1} \times \frac{n}{2n} = \frac{1}{2} = P(R_1)$.

2. Denote by α the common value of P{High | input = 1} and P{Low | input = 0}.

(a),(b) The likelihood matrix is shown below on the left and the joint probability matrix on the right.

	Low	High
input = 0	α	$1-\alpha$
input = 1	$1-\alpha$	α

	Low	High
input = 0	$\pi_0\alpha$	$\pi_0(1-\alpha)$
input = 1	$\pi_1(1-\alpha)$	$\pi_1\alpha$

Since $\alpha > 1-\alpha$, the ML decision rule is as indicated by the shaded squares in the likelihood matrix, i.e.,

the ML decision rule is { If output = High, decide that a 1 was being transmitted.
If output = Low, decide that a 0 was being transmitted.

If input = 0, the ML decision is in error if output = High, which occurs with probability $1-\alpha$.
If input = 1, the ML decision is in error if output = Low, which occurs with probability $1-\alpha$.
Hence, $P_{e,ML} = (1-\alpha)\pi_0 + (1-\alpha)\pi_1 = (1-\alpha)$.

(b) $\pi_0\alpha > \pi_1(1-\alpha)$ and $\pi_0(1-\alpha) > \pi_1\alpha$ for the given data and hence the MAP rule is as indicated above by shading in the joint probability matrix: Always decide that the input was a 0. It follows that the decision is in error only when a 1 is transmitted. Thus, $P_{e,MAP} = \pi_1 < 1-\alpha$.

(c) The ML and MAP decision rules are the same if and only if the following two conditions hold:

$\pi_0\alpha > \pi_1(1-\alpha) \Leftrightarrow \pi_1 < \alpha$,

and $\pi_1\alpha > \pi_0(1-\alpha) \Leftrightarrow \pi_0 < \alpha$, which together imply that $1-\alpha < \pi_0 < \alpha$.

3. All three probabilities can be computed from the given information:

$P(A|B) = P(B|A)P(A)/P(B)$. $P(A^c \quad B^c) = 1 - P(A \cap B) = 1 - P(B|A)P(A)$ and

$P(B^c|A^c) = 1 - P(B|A^c) = 1 - P(A^c \cap B)/P(A^c) = 1 - \frac{P(B) - P(A \cap B)}{1 - P(A)}$.

4. Let $(1+\alpha)/2$ and $(1-\alpha)/2$ denote the probabilities of winning the first and third sets. There are six possible outcomes as shown in the table below where the probabilities are also given.

Outcome	**Probability**	**Win Match?**	**Value of X**	**Value of Y**
WW	$(1+\alpha)/4$	Y	1	2
WLW	$(1-\alpha^2)/8$	Y	1	3
LWW	$(1-\alpha)^2/8$	Y	2	3
LL	$(1-\alpha)/4$	N	0	2
LWL	$(1-\alpha^2)/8$	N	2	3
WLL	$(1+\alpha)^2/8$	N	1	3

(a) Hence, $P(\text{win match}) = \frac{2 + 2\alpha + 1 - \alpha^2 + 1 - 2\alpha + \alpha^2}{8} = \frac{1}{2}$.

(b) $E[\mathbf{X}] = 1\times[(1+\alpha)/4 + (1-\alpha^2)/8 + (1+\alpha)^2/8] + 2\times[(1-\alpha^2)/8 + (1-\alpha)^2/8] = 1$.

(c) $E[\mathbf{Y}] = 2\times[(1+\alpha)/4 + (1-\alpha)/4] + 3\times[1 - \{(1+\alpha)/4 + (1-\alpha)/4\}] = 2\times(1/2) + 3\times(1/2) = 5/2 = 2.5$.

Note that the answers do not depend on the value of α.