University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 313

Solutions to Hour Exam # 2

Problem 1 (26 points)

Bubba rolls a fair 6-faced die till he rolls the number 6 for the second time. Let $\mathcal{S}$ be a random variable that denotes the number of times he rolls the die.

a. Find $P\{\mathcal{S} > 6\}$.

Answer: If $\mathcal{S} > 6$ this implies that in the first 6 rolls there is at most one 6, i.e., either **no** sixes or **one** six in the first 6 rolls. These two events have probabilities $(1-p)^6$ and $\binom{6}{1}\, p(1-p)^5$, respectively, where $p = \frac{1}{6}$ is the probability of rolling a 6.

Therefore $P\{\mathcal{S} > 6\} = (1-p)^6 + \binom{6}{1}\, p(1-p)^5$, which in this case $=$ 0.7368.

b. Find the probability mass function of $\mathcal{S}$, i.e., find $p_{\mathcal{S}}(k) = P\{\mathcal{S} = k\}$. *Specify $p_{\mathcal{S}}(k)$ for all values of k.*

Answer: Observe that if $\mathcal{S} = k$, the k^{th} roll resulted in a 6, and **one** of the previous $k-1$ rolls also resulted in a 6. Therefore for $k \geq 2$, the probability of this event is

$$P\{\mathcal{S} = k\} = \binom{k-1}{1} p(1-p)^{k-2} \cdot p = \begin{cases} (k-1)\, p^2\, (1-p)^{k-2}, & k = 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases}$$

c. If Bubba gets 2 sixes within 10 rolls, he gets $\dfrac{10}{\mathcal{S}-1}$ points, otherwise he gets 0 points.

For example, if he gets 2 sixes in 7 rolls, then he gets $10/6$ points; on the other hand, if he gets 2 sixes in 11 rolls, he gets 0 points.
Find the expected number of points that he gets.

Answer: Let Y be a random variable denoting the number of points Bubba gets. If $\mathcal{S} > 10$, $Y = 0$; and if $2 \leq \mathcal{S} \leq 10$, $Y = \frac{10}{\mathcal{S}-1}$. Let these 10 values of Y be $\{y_0, y_1, \ldots, y_9\}$, where $y_0 = 0, y_1 = 10, y_2 = 5, \ldots, y_9 = 10/9$. Then

$$E[Y] = \sum_{j=0}^{9} y_j P(Y = y_j) = \sum_{k=2}^{10} \frac{10}{k-1} \cdot (k-1)\, p^2 (1-p)^{k-2}$$

$$= \frac{10p^2\,(1-(1-p)^9)}{1-(1-p)} = 10p\,\left(1-(1-p)^9\right),$$

which in this case is equal to $10\left(\frac{1}{6}\right)\left(1-\left(\frac{5}{6}\right)^9\right) = 1.3436$.

Problem 2 (22 points)

X is a continuous random variable with probability density function given by $f_X(u) = e^{-\pi u^2}$, for all $u \in \mathbb{R}$. Define a new random variable Y by $Y = (1-\sqrt{\pi}X)^2$.

a. Find the expected value of Y.

Answer: The essence of this problem is to realize that X is a Gaussian random variable with zero-mean and variance $\frac{1}{2\pi}$. Therefore, $E[X^2] = Var(X)+(E[X])^2 = Var(X) = \frac{1}{2\pi}$. Now

$$\begin{aligned} E[Y] &= E[1-2\sqrt{\pi}X+\pi X^2] = 1-2\sqrt{\pi}E[X]+\pi E[X^2] \\ &= 1-0+\frac{1}{2\pi} = 1+\frac{1}{2\pi} \end{aligned}$$

b. What is the probability density function of Y, $f_Y(v)$? *Specify this for all $v \in \mathbb{R}$.*

Answer: To find $f_Y(v), \forall v \in \mathbb{R}$, we first find the CDF of Y. For $v \geq 0$ we have

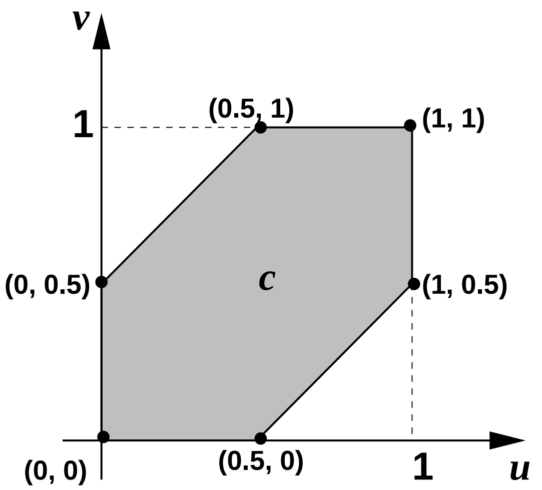
$$\begin{aligned} F_Y(v) &= P\{Y \leq v\} = P\{(1-\sqrt{\pi}X)^2 \leq v\} = P\{(\sqrt{\pi}X-1)^2 \leq v\} \\ &= P\{-\sqrt{v} \leq \sqrt{\pi}X - 1 \leq \sqrt{v}\} = P\{\frac{-\sqrt{v}+1}{\sqrt{\pi}} \leq X \leq \frac{\sqrt{v}+1}{\sqrt{\pi}}\} \\ &= F_X\left(\frac{\sqrt{v}+1}{\sqrt{\pi}}\right) - F_X\left(\frac{-\sqrt{v}+1}{\sqrt{\pi}}\right) \end{aligned}$$

For $v < 0, F_Y(v) = 0$. Therefore, the probability density function of Y, which is given by $f_Y(v) = \frac{d}{dv}F_Y(v)$, is 0 for $v < 0$, and for $v \geq 0$

$$f_Y(v) = \frac{1}{2\sqrt{\pi v}}\left[f_X\left(\frac{\sqrt{v}+1}{\sqrt{\pi}}\right) + f_X\left(\frac{-\sqrt{v}+1}{\sqrt{\pi}}\right)\right] = \frac{1}{2\sqrt{\pi v}}\left(e^{-\pi(\sqrt{v}+1)^2} + e^{-\pi(1-\sqrt{v})^2}\right).$$

Problem 3 (32 points)

Random variables X and Y are jointly continuous with a joint probability density function that is a constant c over the hexagon shown in the figure below.



a. What is the value of c?

Answer: The area of the hexagon in the above figure is $1 - 2 \cdot \left(\frac{1}{2} \cdot 0.5 \cdot 0.5\right) = 3/4$. Therefore, $c = \dfrac{1}{3/4} = \dfrac{4}{3}$.

b. Find the marginal probability density function $f_Y(a)$ (the marginal pdf of Y), for all $a \in \mathbb{R}$. *Specify the pdf for all values of* a.

Answer: For $a < 0$, $f_Y(a) = 0$.
For $0 \leq a < 0.5$,

$$f_Y(a) = \int_{u=0}^{a+0.5} \frac{4}{3}\, du = \frac{4}{3}(a + 0.5)$$

For $0.5 \leq a \leq 1$,

$$f_Y(a) = \int_{u=v-0.5}^{1} \frac{4}{3}\, du = \frac{4}{3}(1.5 - a)$$

c. Find the value of the joint cumulative distribution function at the point $\left(\dfrac{3}{4}, \dfrac{3}{4}\right)$.

Answer: This can again be obtained using simple geometry.
$F_{X,Y}\left(\frac{3}{4}, \frac{3}{4}\right) = \frac{4}{3}\left(\left(\frac{3}{4}\right)^2 - 2 \cdot \frac{1}{2}\left(\frac{1}{4}\right)^2\right) = \dfrac{2}{3}$.

d. Calculate the probability that $\left(X - \dfrac{1}{2}\right)^2 + \left(Y - \dfrac{1}{2}\right)^2 \leq \dfrac{1}{4}$.

Answer: The above equation describes a circle of radius $1/2$ centered at $\left(\frac{1}{2}, \frac{1}{2}\right)$. The area of this circle is $\pi/4$. We now use geometry to find the volume under the joint pdf in the desired area (which is **part** of the above circle).
$P\left\{\left(X - \frac{1}{2}\right)^2 + \left(Y - \frac{1}{2}\right)^2 \leq \frac{1}{4}\right\} = \frac{4}{3}\left[\frac{1}{2} + \left(\frac{1}{2}\right)\left(\frac{\pi}{4} - \frac{1}{2}\right)\right] = \frac{\pi}{6} + \frac{1}{3}$.

Problem 4 (20 points)

The number of arrivals at a bank is a Poisson random variable with an average rate of 20 customers/hour.

a. Find the probability that 5 customers enter the bank in a 15 minute interval.

Answer: Since the rate of the Poisson random variable is 20 customers/hour, the number of customers entering in a 15-minute interval is a Poisson random variable X with $\lambda = 20/4 = 5$.
Therefore, $P\{X = 5\} = \dfrac{e^{-5}\, 5^5}{5!} = 0.175$.

b. Given that 20 customers entered the bank in 1 hour, what is the probability that 5 of them entered in the **first** 15 minutes?

Answer: If 20 customers entered the bank in 1 hour and 5 of them entered in the first 15 minutes, this implies that 15 entered in the last 45 minutes. Since we're dealing with a Poisson random variable, the event {5 people enter in the first 15 minutes} is independent of the event {15 people enter in the next 45 minutes}. Therefore

$$P(5 \text{ in first 15 minutes} \mid 20 \text{ in one hour}) = \frac{P(5 \text{ in first 15 minutes }, \ 15 \text{ in next 45 minutes })}{P(\ 20 \text{ in one hour})}$$

$$= \frac{\frac{e^{-5}\,5^5}{5!} \cdot \frac{e^{-15}\,15^{15}}{15!}}{\frac{e^{-20}\,20^{20}}{20!}} = \binom{20}{5}\left(\frac{5}{20}\right)^5\left(\frac{15}{20}\right)^{15} = \binom{20}{5}\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^{15}$$

Thus we see that this is a binomial random variable. In fact, given that 20 customers entered the bank in 1 hour, their times of arrival are all **independent** random variables.

c. Extra Credit [20 pts]: Given that **only** 1 person entered the bank in one hour, find the probability density function of her **time of entry**, denoted by the random variable $\mathcal{T}$.

Answer: Let X be the number of people that enter the bank in one hour. We will first find the CDF of $\mathcal{T}$ given that $X = 1$, $P(\mathcal{T} \leq u|X = 1)$. This is the probability that exactly one person entered the bank within time u and **zero** people entered in the remaining $1 - u$, given that exactly one person entered the bank in one hour. Therefore

$$F_{\mathcal{T}}(u|X = 1) = P(\mathcal{T} \leq u|X = 1) = \frac{\frac{e^{-20u}\,(20u)}{1!} \cdot \frac{e^{-20(1-u)}\,(20(1-u))^0}{0!}}{\frac{e^{-20}\,20^1}{1!}} = u, \quad 0 \leq u \leq 1,$$

and $F_{\mathcal{T}}(u|X = 1) = 1$ if $u > 1$ and 0 if $u < 0$. Therefore we see that the conditional pdf of $\mathcal{T}$ is uniform over the interval $[0, 1]$.