University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 313

Solutions to Hour Exam # 1

Problem 1 (24 points)

The following are statements about events A, B, and C with nonzero probabilities $P(A)$, $P(B)$, and $P(C)$. The events are arbitrary, unless a specific condition is specified in the statement. The random variable X is also arbitrary, unless otherwise specified in the statement. The function $F_X(u)$ is the cumulative distribution function (CDF) of X.
Answer True if the statement is true for **all** such events A, B, C, and random variables X. Otherwise, answer False.

True	False	
☐	☑	If A and B are independent, then AB^c and A^cB are independent.
☑	☐	if the events A, B, C are disjoint and $P(A) + P(B) + P(C) = 1$, then exactly five of the eight regions in the Karnaugh map for A, B, C have zero probability
☐	☑	$P\Big((A \oplus B)\vert A\Big) = P(B^c)$
☐	☑	If $P(A\vert B) = P(B\vert A)$, then $P(A) = P(B)$
☑	☐	$P(AB) \leq \sqrt{P(A)P(B)}$
☐	☑	If A and B are disjoint, then $P(A\vert B) + P(A\vert B^c) = 1$
☑	☐	If $P\{X = \pi\} = P\{X = -\pi\} = \frac{1}{2}$, then $\sin(X)$ is a discrete random variable
☐	☑	If $a < b$ then $F_X(a) < F_X(b)$

Problem 2 (24 points)

Dilbert has 3 coins in his pocket, 2 of which are fair, and 1 which is biased. The probability that a coin picked at random from his pocket will land Tails when it is tossed is $7/12$.

a. What is the probability that the biased coin will fall Tails on a given toss?

Answer: Define the following events:

$$\begin{aligned} \mathcal{B} &= \texttt{Dilbert picks the biased coin} \\ \mathcal{T} &= \texttt{the coin chosen lands Tails} \end{aligned}$$

Then, by the theorem of total probability, we have:

$$P(\mathcal{T}) = P(\mathcal{T}|\mathcal{B})P(\mathcal{B}) + P(\mathcal{T}|\mathcal{B}^c)P(\mathcal{B}^c)$$
$$\Longrightarrow \frac{7}{12} = P(\mathcal{T}|\mathcal{B})\frac{1}{3} + \frac{1}{2}\cdot\frac{2}{3} \Longrightarrow P(\mathcal{T}|\mathcal{B}) = \frac{3}{4}$$

$$\boxed{P(\text{Tails}) \;=\; \frac{3}{4}}$$

b. If Dilbert picks **two** coins out of his pocket what is the probability that he did **not** pick the biased coin?

Answer: This part is quite straightforward. Let $\mathcal{B}$ denote the event that a biased coin is picked,

$$P(\mathcal{B}^c) = \frac{2}{3}\cdot\frac{1}{2} = \frac{1}{3}$$

You can also think of this as $\binom{2}{2}/\binom{3}{2}$ — because $\binom{2}{2}$ is the number of ways of choosing both fair coins, and $\binom{3}{2}$ is the number of ways of choosing 2 coins out of 3.

$$\boxed{P(\mathcal{B}^c) \;=\; \frac{1}{3}}$$

c. If he picks **two** coins out of his pocket, tosses each one **once**, and observes a Head and a Tail, what is the (conditional) probability that he did **not** pick the biased coin?

Answer: With two coins there are basically 4 possible outcomes — $\{HH, HT, TH, TT\}$. If the two coins are fair, the probability of getting a Head and a Tail is $1/2$, i.e., $P(1\,H\ \&\ 1\,T|\mathcal{B}^c) = 1/2$. We only need to find $P(1\,H\ \&\ 1\,T|\mathcal{B})$, i.e., the probability that we get 1 Head and 1 Tail if one of the coins chosen is biased. If the event $\mathcal{B}$ occurs, then the biased coin could fall Heads, or it could fall Tails. Accordingly,

$$P(1\,H\ \&\ 1\,T|\mathcal{B}) = \frac{1}{4}\cdot\frac{1}{2} + \frac{3}{4}\cdot\frac{1}{2} = \frac{1}{2} \quad \text{(are you surprised?)}$$

We are now in a position to find $P(\mathcal{B}^c|1\,H\ \&\ 1\,T)$ using Bayes' rule:

$$P(\mathcal{B}^c|1\,H\ \&\ 1\,T) = \frac{P(1\,H\ \&\ 1\,T|\mathcal{B}^c)\cdot P(\mathcal{B}^c)}{P(1\,H\ \&\ 1\,T)} \tag{1}$$

We will calculate $P(1\,H\ \&\ 1\,T)$ using the theorem of total probability: $P(1\,H\ \&\ 1\,T) = P(1\,H\ \&\ 1\,T|\mathcal{B})P(\mathcal{B}) + P(1\,H\ \&\ 1\,T|\mathcal{B}^c)P(\mathcal{B}^c) = (1/2)(2/3) + (1/2)(1/3) = 1/2$. Thus, we see that the event $\{1\,H\ \&\ 1\,T\}$ is independent of $\mathcal{B}$! Substituting this value back in Eq.(1) we get

$$P(\mathcal{B}^c|1\,H\ \&\ 1\,T) = \frac{(1/2)\,(1/3)}{(1/2)} = \frac{1}{3}$$

$$\boxed{P\Big((\text{Biased coin})^c|1 \text{ Head } \&\ 1 \text{ Tail}\Big) \;=\; \frac{1}{3}}$$

Problem 3 (32 points)

Toejam Inc., a new shoe store, has a special opening offer this week. They have 13 different pairs of shoes in a barrel at the door. A customer pays $\$1$ for the chance of picking out one shoe at random from the barrel. You and your friend walk into the store and decide to pick **two** shoes **each** in the hopes of getting a matching pair.

a. If you pick first, what is the probability of your getting a matching pair of shoes?

Answer: The first shoe can be picked in any of the 26 possible ways. The second shoe **has to be** one out of the remaining 25 shoes. Therefore the probability of getting a matching pair is $1 \cdot 1/25$.

$$\boxed{P(\text{matching pair}) \;=\; \frac{1}{25}}$$

b. Your friend picks next. What is the probability that she gets a matching pair of shoes?

Answer: The answer is the same as before by symmetry! We can think of the situation as the friend getting one of the 13 pairs and the remaining shoes are "don't cares" — just like we considered when we calculated the probability of you getting a pair. To convince ourselves, let us use the theorem of total probability. Define the following events:

$$\mathcal{M} = \texttt{you pick a matching pair}$$
$$\mathcal{N} = \texttt{your friend picks a matching pair}$$

Then, by the TTP

$$P(\mathcal{N}) = P(\mathcal{N}|\mathcal{M})P(\mathcal{M}) + P(\mathcal{N}|\mathcal{M}^c)P(\mathcal{M}^c) = \frac{24}{24} \cdot \frac{1}{23} \cdot \frac{1}{25} + \frac{22}{24} \cdot \frac{1}{23} \cdot \frac{24}{25} = \frac{1}{25},$$

where $22/24$ is the probability of picking a shoe from one of the 11 pairs that have "mates" and $1/23$ is the probability of picking the mate of the shoe that she picked first. Therefore

$$\boxed{P(\text{matching pair}) \;=\; \frac{1}{25}}$$

c. What is the probability that there is at least one matching pair of shoes in the 4 that have been picked?

Answer: If $\mathcal{P}$ is the event that at least one pair of shoes is picked among the 4 shoes picked, then $\mathcal{P}^c$ is the event that **no** pair is picked. The size of the sample space is $\binom{26}{4}$ — the number of ways of choosing 4 shoes out of 26. For the number of outcomes in $\mathcal{P}^c$ we now have to choose 4 shoes from 4 different pairs, and the number of ways of doing that is $\binom{13}{4} 2^4$. So,

$$P(\mathcal{P}) = 1 - P(\mathcal{P}^c) = 1 - \frac{\binom{13}{4} 2^4}{\binom{26}{4}} = 1 - \frac{88}{115} = \frac{27}{115}.$$

$$P(\text{at least 1 matching pair}) = \frac{27}{115}$$

d. If N is a random variable denoting the number of matching pairs in the 4 shoes that have been picked, find the probability mass function (pmf) of N, i.e., find $P\{N = u\}$ for all $u \in \mathbb{R}$. *To obtain full credit you have to specify the pmf of N for* **all** *real numbers $u \in \mathbb{R}$.*

Answer: First note that N takes on values $0, 1$ or 2. The value of the pmf at $u = 0$ was found in part (c):

$$P\{N = 0\} = \frac{\binom{13}{4} 2^4}{\binom{26}{4}} = \frac{88}{115}.$$

The probability that there are exactly 2 pairs of shoes in the 4 picked is not hard to find also. The number of ways this can happen is just $\binom{13}{2}$. Therefore

$$P\{N = 2\} = \frac{\binom{13}{2}}{\binom{26}{4}} = \frac{3}{575}.$$

From this and part (c), we have $P\{N = 1\} = 27/115 - 3/575 = 132/575$. We can also calculate $P\{N = 1\}$ separately: the number of ways of getting exactly 1 pair of matching shoes out of the 4 shoes chosen is the number of ways of choosing one matching pair out of 13 "times" the number of ways of picking the other 2 shoes out of the remaining 12 pairs (each shoe from a different pair). Therefore

$$P\{N = 1\} = \frac{\binom{13}{1}\binom{12}{2} 2^2}{\binom{26}{4}} = \frac{132}{575}.$$

$$p_N(u) = \begin{cases} 88/115, & u = 0 \\ 132/575, & u = 1 \\ 3/575, & u = 2 \\ 0, & \text{otherwise} \end{cases}$$

Problem 4 (20 points)

Y is a **continuous** random variable. Its cumulative distribution function (CDF) is defined as follows:

$$F_Y(u) = \begin{cases} 0, & u < -4 \\ \dfrac{u+4}{6}, & -4 \le u < -2 \\ \alpha\, u^2 + \beta\, u + \dfrac{1}{2}, & -2 \le u < 2 \\ 1, & u \ge 2 \end{cases}$$

a. What is the probability that $|Y| \ge 3$?

Answer: We get the answer to this part by just reading values off the given CDF:

$$P\{|Y| \ge 3\} = P\{Y \ge 3\} + P\{Y \le -3\} = 1 - F_Y(3) + F_Y(-3) = \frac{1}{6}$$

$$\boxed{P\{|Y| \ge 3\} = \frac{1}{6}}$$

b. Find the values of the real numbers α and β.

Answer: Since the random variable Y is continuous, the CDF $F_Y(u)$ has to be continuous. Now$F_Y(-2) = \dfrac{-2+4}{6} = 1/3$ and $F_Y(2) = 1$; therefore from these two equations, we get two simultaneous equations for α and β:

$$\begin{aligned} \alpha\,(-2)^2 + \beta\,(-2) + 1/2 &= 1/3 \\ \alpha\,(2)^2 + \beta\,(2) + 1/2 &= 1 \end{aligned}$$

from which we get $\alpha = 1/24$ and $\beta = 1/6$.

$$\boxed{\alpha = \frac{1}{24}} \qquad \boxed{\beta = \frac{1}{6}}$$

c. Calculate the probability $P\{Y^2 - 1 < 0\}$.

Answer: Again, reading values off the CDF, we get:

$$\begin{aligned} P\{Y^2 - 1 < 0\} &= P\{-1 < Y < 1\} = F_Y(1) - F_Y(-1) \\ &= (\alpha + \beta + 1/2) - (\alpha - \beta + 1/2) = 2\beta = 1/3 \end{aligned}$$

$$\boxed{P\{Y^2 - 1 < 0\} = \frac{1}{3}}$$