University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 313

Solutions to Hour Exam # 1

Problem 1 (24 points)

The following are statements about events A, B, and C with nonzero probabilities $P(A)$, $P(B)$, and $P(C)$. The events are arbitrary, unless a specific condition is specified in the statement. Answer True if the statement is true for **all** such events A, B, C. Otherwise, answer False.

True	False	
☐	☑	if $A \subset B$, then $P(B \mid A^c) = 0$
☑	☐	$P(A \oplus B) = P(A \mid B^c)P(B^c) + P(A^c \mid B)P(B)$
☐	☑	$P(AB) + P(AC) + P(BC) - P(ABC) = P(ABC^c) + P(AB^cC) + P(A^cBC)$
☑	☐	if $P(A) = P(B)$, then $P(A \mid B) = P(B \mid A)$
☑	☐	if $P(A \mid B) > P(B \mid A)$, then $P(A) > P(B)$
☐	☑	if $P(A \mid B) = P(B \mid A)$, then A and B are independent
☐	☑	if $P(ABC) = P(A)P(B)P(C)$, then A, B, C are independent events
☑	☐	if A, B, C are independent, then $P(A \cup B \cup C) + P(A^c)P(B^c)P(C^c) = 1$

Problem 2 (17 points)

A student is taking a multiple choice exam with 4 alternatives for each question. If the student knows the answer, she selects it (and is always right). Otherwise, she selects at random from the 4 possible answers. Suppose that the student knows the answers to 70% of the questions.

a. What is the probability that she selects the correct answer to a given question?

Answer: Define the following events:

$$\begin{aligned} \mathcal{C} &= \texttt{she picks the correct answer} \\ \mathcal{K} &= \texttt{she knows the correct answer} \end{aligned}$$

Then, by the theorem of total probability, we have:

$$P(\mathcal{C}) = P(\mathcal{C} \mid \mathcal{K})P(\mathcal{K}) + P(\mathcal{C} \mid \mathcal{K}^c)P(\mathcal{K}^c) = 1(0.7) + (0.25)(0.3) = 0.775$$

$$\boxed{P(\text{correct}) = 0.775}$$

b. If the student selects the correct answer, what is the conditional probability that she actually knows the answer?

Answer: We can use the Bayes inversion formula, to compute $P(\mathcal{K}|\mathcal{C})$ as follows:

$$P(\mathcal{K}|\mathcal{C}) = \frac{P(\mathcal{C}|\mathcal{K})\,P(\mathcal{K})}{P(\mathcal{C})} = \frac{1(0.7)}{0.775} = \frac{28}{31}$$

$$\boxed{P(\text{knows}|\text{correct}) = \frac{28}{31}}$$

Problem 3 (32 points)

Alice and Bob play the following game of dice. Each round of the game consists of Alice rolling one die and Bob rolling two dice. All the dice are fair. Here are the rules of this game:

- If Alice's number falls strictly between the two numbers rolled by Bob, then Alice wins and the game stops;
- If Alice's number is equal to one of the numbers rolled by Bob, then it is a tie and the round is repeated until there is a winner;
- In all other cases, Bob wins and the game stops.

For example: if Alice rolls (3) while Bob rolls $(1,4)$ or $(4,1)$, then Alice wins; if Alice rolls (3) and Bob rolls $(1,3)$, then there is a tie; if Alice rolls (3) while Bob rolls $(1,1)$, then Bob wins. Based on these rules, what is the probability that:

a. Alice wins on the first round, given that she rolls a (3)?

Answer: Alice wins when she rolls (3) if and only if Bob rolls one of the following:

$$\left\{ \begin{array}{ll} (1,4),(1,5),(1,6), & (2,4),(2,5),(2,6) \\ (4,1),(5,1),(6,1), & (4,2),(5,2),(6,2) \end{array} \right\}$$

Thus there are 12 outcomes for Bob which lead to a win by Alice, out of the total of 36 equally likely outcomes that are possible. Therefore $P(A|3) = 12/36 = 1/3$.

$$\boxed{P(A|3) = \frac{1}{3}}$$

b. There is a tie on the first round?

Alice wins on the first round?

Bob wins on the first round?

Answer: Let A be the event that Alice wins on a given round, say the first round. To find $P(A)$, we will condition on the outcome of Alice's roll. It is obvious that Alice cannot win if she rolls a (1) or a (6). In a manner similar to part (**a**), we see that if Alice rolls a (2), (4), or (5), the number of outcomes for Bob that lead to a win by Alice is 8, 12, and 8, respectively. Therefore, by the theorem of total probability, we have

$$P(A) = \sum_{i=1}^{6} P(A|\texttt{Alice rolls } i)\, P(\texttt{Alice rolls } i) = \frac{1}{6} \cdot \left(\frac{8+12+12+8}{36}\right) = \frac{5}{27}$$

Now define the event $T = \texttt{tie occurs on the first round}$. Given that Alice rolls (i), a tie occurs if either of the dice in Bob's roll shows (i). Therefore:

$$P(T|\texttt{Alice rolls } i) = P(\texttt{die \#1 = } i) + P(\texttt{die \#2 = } i) - P(\texttt{both dice = } i)$$

$$= \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}.$$

Hence, by the theorem of total probability, we obtain:

$$P(T) = \sum_{i=1}^{6} P(T|\texttt{Alice rolls } i)\, P(\texttt{Alice rolls } i) = \sum_{i=1}^{6} \frac{11}{36} \cdot \frac{1}{6} = \frac{11}{36}$$

Finally, the easiest way to find the probability of the event B that Bob wins on the first round is as follows:

$$P(B) = 1 - P(A) - P(T) = 1 - \frac{11}{36} - \frac{5}{27} = \frac{55}{108}$$

This has to be so, as the events A, B, and T are disjoint and one of them always occurs.

$$\boxed{P(A) = \frac{5}{27}} \quad \boxed{P(B) = \frac{55}{108}} \quad \boxed{P(T) = \frac{11}{36}}$$

c. Alice wins the game?

Answer: Define the event $A_k = \texttt{Alice wins the game in exactly } k \texttt{ rounds}$. Then:

$$P(A_k) = P(A)\,[P(T)]^{k-1}$$

This is so because Alice wins the game in exactly k rounds if and only if Alice wins on the k-th round and the previous $k-1$ rounds all resulted in a tie. Therefore, the overall probability that Alice wins the game is given by:

$$P(\texttt{Alice wins}) = P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(A)\,[P(T)]^{k-1} = \frac{P(A)}{1 - P(T)}$$

$$= \frac{5/27}{25/36} = \frac{4}{15}$$

$$\boxed{P(\text{Alice wins}) = \frac{4}{15}}$$

Problem 4 (27 points)

Alice, Bob, and Carol attend a dinner party together with $n-3$ other people. Upon entering, all the diners hand over their business cards to the doorman for drawing in a lottery. There are **two prizes** in the lottery. Thus after the dinner is over, two cards are drawn at random (one after the other, without replacement) and their bearers each win a prize.

a. What is the probability that Alice wins a prize?

Answer: Define the event A_i = `Alice wins a prize at drawing number` i, where $i = 1, 2$. hen using the theorem of total probability,

$$P(\texttt{Alice wins a prize}) = P(\mathcal{A}|A_1)\,P(A_1) + P(\mathcal{A}|A_1^c)\,P(A_1^c)$$

$$= 1 \cdot \frac{1}{n} + \frac{1}{n-1} \cdot \frac{n-1}{n} = \frac{2}{n}$$

$$\boxed{P(\mathcal{A}) = \frac{2}{n}}$$

Next week Alice, Bob, and Carol attend this dinner party again. There are again $n-3$ other people at this party, and exactly the same lottery is supposed to take place. However, this time around Bob and Carol find out, **just after the first card is drawn**, that the doorman forgot to put their cards in the draw-basket. After much argument, the lottery organizers agree to put **two cards each** for Bob and Carol in the draw-basket before the second card is drawn.

b. What is the probability that Bob wins a prize?

Answer: The number of cards in the basket is now $n+1$, consisting of the $n-2$ cards that were put in the basket in the first place, minus the one card that was already withdrawn, plus the 4 cards that belong to Bob and Carol. Therefore:

$$P(\texttt{Bob wins a prize}) = \frac{2}{n+1}$$

$$\boxed{P(\mathcal{B}) = \frac{2}{n+1}}$$

c. What is the probability that Alice wins a prize?

Answer: The probability that Alice wins a prize is given by the same total-probability expression as in part (**a**), but the probabilities of various events now change as follows:

$$P(\texttt{Alice wins a prize}) = P(\mathcal{A}|A_1)\,P(A_1) + P(\mathcal{A}|A_1^c)\,P(A_1^c)$$

$$= 1 \cdot \frac{1}{n-2} + \frac{1}{n+1} \cdot \frac{n-3}{n-2} = \frac{2}{n+1} \cdot \frac{n-1}{n-2}$$

$$\boxed{P(\mathcal{A}) = \frac{2}{n+1} \cdot \frac{n-1}{n-2}}$$

Who has better chances of winning a prize: Alice or Bob?

Answer: Since $(n-1)/(n-2) > 1$, is is clear that Alice has better chances.

☑ Alice ☐ Bob ☐ same

Extra-credit (20 points): Just before Bob and Carol put cards in the draw-basket, an engineering student from U of I walks up and says that she knows how to fix the problem in a jiffy. By making a simple modification of the lottery rules and procedures, that only takes seconds to implement, she equalizes all the diners' probabilities of winning a prize.

What modification did she propose?

Answer: The key here is to realize that if all the diners' probabilities are equalized, then that equal probability of winning a prize has to be exactly $2/n$ for everbody, just as in part (**a**).

Here is one simple way to achieve this. Draw one card at random just before Bob and Carol put their cards in the draw-basket and **throw it away**. The number of cards in the basket is now $n-4$. Now let Bob and Carol put in the basket two cards each, thereby making the total number of cards in the basket exactly n. Once this is done, draw a card to decide the second winner. Let us analyze the probability that Carol (or Bob) wins a prize, as compared to the probability that Alice (who represents one of the other $n-2$ diners) wins a prize.

$$P(\texttt{Alice wins a prize}) = 1 \cdot \frac{1}{n-2} + \frac{n-3}{n-2} \cdot \frac{n-4}{n-3} \cdot \frac{1}{n} = \frac{2}{n}$$

where:

$$\frac{1}{n-2} = \text{the probability of winning a prize on the first draw}$$

$$\frac{n-3}{n-4} = \text{the probability of } \textbf{not} \text{ winning a prize on the first draw}$$

$$\frac{n-4}{n-3} = \text{the probability of } \textbf{not} \text{ getting thrown out on the second draw}$$

$$\frac{1}{n} = \text{the probability of winning a prize on the third draw}$$

Now $P(\texttt{Carol wins a prize}) = 2/n$, because there are now n cards altogether and two of them belong to Carol. Therefore all the diners have equal probabilities of winning.

Pretty sharp U of I engineering student, eh?