University of Illinois Fall 1998

ECE 313: Solutions to Problem Set #2

1. **(a)** One easy way to arrive at the total number of subsets (events) that can be formed from a set (sample space) of size n is using BPOC: Let the sample space be denoted by

$$\Omega = \{x_1, x_2, \ldots, x_n\}$$

For each subset you can associate an n-tuple formed using the numbers $\{0, 1\}$, where a 1 in the i^{th} position indicates that x_i is in the subset, and a 0 indicates its absence. Thus, if $n = 5$, then the string 10011 denotes a subset $E = \{x_1, x_4, x_5\}$. Now the total number of n-tuples of 0's and 1's that can be formed is precisely $2 \times 2 \times \cdots n(\text{times}) \times 2 = 2^n$.

The total number of events can be broken down as $\sum_{i=0}^{n} \left\{ \begin{array}{c} \texttt{number of events} \\ \texttt{with } i \texttt{ outcomes} \end{array} \right\}$. The number of distinct events with i outcomes is $\binom{n}{i}$. Therefore $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

(b) Let η_{even} denote the number of events containing an even number of outcomes, and let η_{odd} denote the number of events containing an odd number of outcomes. The number of events with exactly i outcomes is $\binom{n}{i}$. Thus

$$\eta_{\text{even}} = \sum_{\substack{i=0 \\ i \text{ even}}}^{n} \binom{n}{i} \qquad \text{and} \qquad \eta_{\text{odd}} = \sum_{\substack{i=0 \\ i \text{ odd}}}^{n} \binom{n}{i}$$

We now use the binomial theorem in two differents ways to establish useful relations between η_{even} and η_{odd}. First, we have

$$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot 1^i \cdot 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i} = \sum_{\substack{i=0 \\ i \text{ even}}}^{n} \binom{n}{i} + \sum_{\substack{i=0 \\ i \text{ odd}}}^{n} \binom{n}{i} = \eta_{\text{even}} + \eta_{\text{odd}}$$

This gives one equation in the two unknowns η_{even} and η_{odd}. Here is how we can use the binomial theorem to get the second relation between η_{even} and η_{odd}.

$$0 = (1-1)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot (-1)^i \cdot 1^{n-i} = \sum_{\substack{i=0 \\ i \text{ even}}}^{n} \binom{n}{i} \cdot (+1) + \sum_{\substack{i=0 \\ i \text{ odd}}}^{n} \binom{n}{i} \cdot (-1) = \eta_{\text{even}} - \eta_{\text{odd}}$$

It follows from the above that $\eta_{\text{even}} = \eta_{\text{odd}}$. Since $\eta_{\text{even}} + \eta_{\text{odd}} = 2^n$ by the first equation, we further conclude that $\eta_{\text{even}} = \eta_{\text{odd}} = 2^n/2 = 2^{n-1}$, which is what we need to prove.

(c) Pair up each event A with its complement A^c. For example, as a special case, the sure event Ω is paired with the null event $\varnothing$. Notice that we have thus matched up the 2^n events into 2^{n-1} pairs. Now consider the sum of the probabilities of all the 2^n events. Each pair contributes a total of $P(A) + P(A^c) = 1$ to the sum. Since there are 2^{n-1} such pairs, the sum of all the probabilities is 2^{n-1}. Dividing by 2^n, we find that the required "average probability" is 1/2.

Remark: Notice that we have not used the assumption that all the n outcomes are equally likely. This means that *any* assignment of probabilities $p_1, p_2, \ldots, p_n$ to the n outcomes gives the same result that the "average probability" is 1/2, providing only that this assignment satisfies the three axioms of probability measure. This is so because it is always true that $P(A) + P(A^c) = 1$.

(d) Now recall that all the outcomes are equally likely. Thus the probability of an event consisting of i outcomes is i/n. If n is even, then $\binom{n}{n/2}$ events have probability $1/2$. If n is odd, then there are *no* events that occur with probability exactly $1/2$.

2. All you need to solve this problem is to know how to count things — see for instance Chapter 1 of the textbook.

(a) Let b, c, m, s, v denote the five basic flavors. Let A, F, P denote the three toppings. The number of ways of choosing either $0, 1, 2$ or 3 toppings out of 3 is $2^3 = 8$. The number of 'triples' (from three basic flavors) is $\binom{5}{3} = 10$. Therefore, using BPOC, the total number of specialty icecreams $= 8 \times 10 = 80$. Of these, the number of icecreams containing c is $\binom{4}{2} \times 8 = 48$, which is the number of ways to choose two flavors from the set $\{b, m, s, v\}$ multiplied by the number of different toppings possible. Notice that the number of flavor triples containing s is also $\binom{4}{2} = 6$. The number of triples that contain **both** c and s is exactly $\binom{3}{1} = 3$. Hence the number of triples containing either c or s but not both is $6 + 6 - 2 \cdot 3 = 6$. Therefore, the number of icecreams with c or s, but not both $= 6 \times 8 = 48$.

(b) The total number of icecreams that you can form with 3 flavors and 2 toppings is $\binom{5}{3} \times \binom{3}{2} = 10 \times 3 = 30$. Of these, the only combinations that Suzie will **not** eat are s, v, x with A, P toppings, where x stands for one of $\{b, c, m\}$. Therefore there are only 3 out of 30 icecream combos that Suzie will not eat. So she has $30 - 3 = 27$ choices.

(c) A specialty icecream is formed either by using

$\{$`One flavor and at least one topping`$\}$ `OR`
$\{$`Two or more flavors and any number of toppings`$\}$

Therefore, the total number of icecream combos is

$$\binom{5}{1} \cdot (2^3 - 1) + \left(\binom{5}{1} + \binom{5}{1} + \binom{5}{1} + \binom{5}{1}\right) \cdot 2^3 = (2^5 - 1)\,8 - 5 = 243,$$

where $\displaystyle\sum_{i=1}^{5} \binom{5}{i} = 2^5 - 1$, the total number of nonempty subsets of a set with 5 elements.

(d) To choose 3 flavors out of 5, with repetition, we will follow the "donut-rule". This is best pictured as follows: we fix an arrangement of flavors, say b, c, m, s, v. Let a scoop be denoted by a "$\bullet$" — there are 3 of these. We will adopt the convention that k "$\bullet$"s to the right of a letter indicate k scoops of that flavor. So "$b \bullet\bullet c\ m \bullet\ s\ v$" would indicate two scoops of butterscotch, none of chocolate, one scoop of mocha and none of strawberry and vanilla. With this convention, all we have to do is find the different permutations of the letters and $\bullet$'s, with a letter starting the string (a letter **has** to start the string). In this case, all the flavors are treated the same, as are the scoops. So, the total number of choices from the icecreams alone are $\dfrac{(5-1+3)!}{(5-1)!\,3!} = \dbinom{7}{3} = 35$. A similar calculation can be done for choosing 2 toppings out of 3 with replacement to get $\dfrac{(3-1+2)!}{(3-1)!\,2!} = \dbinom{4}{2} = 6$ possibilities. Therefore, the total number of different icecream combos in this scenario is $35 \times 6 = 210$.

(e) **[Extra Credit]**: This part is almost as easy as the previous part once you have sorted out what is going on. Two out of the six scoops are fixed — c and m. We also know that s cannot be one of the scoops. So we have a remaining of 4 scoops to be chosen from 4 flavors. This can be done using the "donut-rule" in $\binom{4+4-1}{4} = \binom{7}{4} = 35$ ways. Now for the toppings: F has to be one of them, so we have 2 left to choose out of 3. This can be done in $\binom{3+2-1}{2} = 6$ ways. Therefore the total number of Scoop-Splurges they can get with their restrictions is $35 \times 6 = 210$.

(**f**) Optional for zero additional credit: Amy's (in Houston), of course!

3. The required expressions are given below, and illustrated with the help of Karnaugh maps in Figure 1.

(a) $A \cup B \cup C$
(b) $(AB^cC^c) \cup (A^cBC^c) \cup (A^cB^cC) \cup (A^cB^cC^c)$
(c) $(A \cup B \cup C)^c = A^cB^cC^c$
(d) ABC
(e) $(AB^cC^c) \cup (A^cBC^c) \cup (A^cB^cC)$
(f) $ABC^c = AB - C$
(g) $A \cup A^cB^c$

4. (**a**) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, we see that $P(A \cup B) \geq P(A)$ and $P(A \cup B) \geq P(B)$. Combining the two bounds gives

$$P(A \cup B) \;\geq\; \max\{P(A), P(B)\} \;=\; \max\{3/4, 1/3\} \;=\; 3/4\ .$$

(**b**) Since $AB \subseteq A$ and $AB \subseteq B$, we conclude that $P(AB) \leq P(A)$ and $P(AB) \leq P(B)$. Combining the two bounds gives

$$P(AB) \;\leq\; \min\{P(A), P(B)\} \;=\; \min\{3/4, 1/3\} \;=\; 1/3\ .$$

Now, using the fact that $P(A \cup B) = P(A) + P(B) - P(AB)$, we have

$$P(AB) \;=\; P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1 \;=\; \frac{3}{4} + \frac{1}{3} - 1 \;=\; \frac{1}{12}$$

since $P(A \cup B) \leq 1$ for all events A and B.

(**c**) Proceeding as in parts (**a**) and (**b**) for $P(A) = 2/3$ and $P(B) = 1/2$, we obtain

$$\begin{aligned} P(A \cup B) &\geq \max\{P(A), P(B)\} = \max\{2/3, 1/2\} = 2/3 \\ P(AB) &\leq \min\{P(A), P(B)\} = \min\{2/3, 1/2\} = 1/2 \\ P(AB) &\geq P(A) + P(B) - 1 = 2/3 + 1/2 - 1 = 1/6 \end{aligned}$$

5. A good advice is to draw a Karnaugh map, at least for parts (**b**) and (**c**), in order to combat the various attempts to confuse you that are evident in this problem.

(**a**) We are given the probabilities of the events $E = A \cup B$ and $F = A^c \cup B^c$, namely $P(E) = 0.6$ and $P(F) = 0.8$. We are asked to find the probability of the event that exactly one of A or B occurred, namely $A \oplus B = (A \cup B) \cap (A^c \cup B^c) = E \cap F$. Note that $E \cup F = (A \cup B) \cup (A^c \cup B^c) = \Omega$, and therefore $P(E \cup F) = 1$. We use the identity $P(E \cup F) = P(E) + P(F) - P(E \cap F)$ to find that $P(A \oplus B) = P(E \cap F) = P(E) + P(F) - P(E \cup F) = 0.6 + 0.8 - 1 = 0.4$.

(**b**) Using DeMorgan laws we get $P(AB) = 1 - P(A^c \cup B^c) = 1 - P(F) = 0.2$ and $P(A^cB^c) = 1 - P(A \cup B) = 1 - P(E) = 0.4$. The new piece of data in part (**b**) is that $C \cap (A^c \cup B^c) = \emptyset$. It follows that $P(A^cB^cC) = P(A^cBC) = P(AB^cC) = 0$. Now, referring to Figure 2, we see that $P(A^cB^cC^c) = P(A^cB^c) - P(A^cBC) = 0.4 - 0 = 0.4$.

The event of interest is $G = C^c \cap (A \oplus B)$, which is denoted by the shaded area in Figure 2. From the figure we have $P(G) = 1 - P(AB) - P(A^cB^c) - P(A^cBC) - P(AB^cC) = 1 - 0.2 - 0.4 - 0 - 0 = 0.4$.

(c) From the new data in part **(b)** we have $P(A^cB^cC) = 0$. Using DeMorgan laws we have $P(A \cup B \cup C^c) = 1 - P(A^cB^cC) = 1$. The following probabilities $P(ABC)$, $P(A)$, $P(C)$, and $P(A^cBC^c)$ cannot be computed.

6. **(a)** A number k is a multiple of both 3 and 4 if and only if it is divisible by their least common multiple. Hence $n = 12$. Similarly, $m = 18$.

(b) We have $A_3 = \{3, 6, 9, \dots\} = \{3\} \cup \{6\} \cup \{9\} \cup \cdots$. Since the events $\{3\}, \{6\}, \{9\}, \cdots$ are all disjoint, it follows that

$$P(A_3) \;=\; \frac{1}{3}\cdot\left(\frac{2}{3}\right)^2 + \frac{1}{3}\cdot\left(\frac{2}{3}\right)^5 + \frac{1}{3}\cdot\left(\frac{2}{3}\right)^8 + \cdots \;=\; \frac{1}{3}\cdot\left(\frac{2}{3}\right)^2 \left[1 + x + x^2 + x^3 + \cdots\right]$$

where $x = \left(\frac{2}{3}\right)^3$. But $1 + x + x^2 + x^3 + \cdots = (1-x)^{-1}$ for $|x| < 1$ (this is our first, but definitely not last, encounter with the geometric series — be warned!). Thus

$$P(A_3) \;=\; \frac{4}{27} \cdot \frac{1}{1 - \frac{8}{27}} \;=\; \frac{4}{19} \;\approx\; 0.210526$$

The approximation 0.210526 above is the unacceptable answer that you would get from your (programmable?) calculator.