University of Illinois

Solutions: Problem Set 13

ECE 313 Fall 1998

1. $f_{X_1}(u) = f_{X_2}(u) = f_{X_3}(u) = 1/2,\ 0 \le u \le 2$, and 0 elsewhere.

   (a) There are 6 (3!) arrangements of $\mathbf{X}_1, \mathbf{X}_2$, and $\mathbf{X}_3$ among themselves, and each of these arrangements is equally likely since $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are i.i.d. Hence, each arrangement has probability 1/6 of occurring — is this reminiscent of Problem 6 in Homework 11? Of these 6 arrangements, $\mathbf{X}_2$ occurs in the middle twice, $(\mathbf{X}_1 < \mathbf{X}_2 < \mathbf{X}_3)$ and $(\mathbf{X}_3 < \mathbf{X}_2 < \mathbf{X}_1)$. Hence the probability that $\mathbf{X}_2$ lies between $\mathbf{X}_1$ and $\mathbf{X}_3$ is $2/6 = 1/3$.

   (b) Let $\mathbf{N}_1, \mathbf{N}_2$ and $\mathbf{N}_3$ denote the smallest, middle and largest, respectively, of the three points $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{X}_3$ (note that $\mathbf{X}_1$ doesn't have to be the smallest). We derived the distribution of $\mathbf{N}_1, \mathbf{N}_2$ and $\mathbf{N}_3$ in the previous homework.

$$F_{N_3}(u) = F_X^3(u) = \begin{cases} 0, & u < 0 \\ \left(\frac{u}{2}\right)^3, & 0 \le u < 2 \\ 1, & u \ge 2 \end{cases}$$

   Therefore $P\{\mathbf{N}_3 > 1\} = 1 - F_{N_3}(1) = 1 - 1/8 = 7/8$.

   (c) Any of the three points can be the largest. Therefore, the desired probability is

$$P\{\mathbf{X}_1 > \mathbf{X}_2 + \mathbf{X}_3\} + P\{\mathbf{X}_2 > \mathbf{X}_1 + \mathbf{X}_3\} + P\{\mathbf{X}_3 > \mathbf{X}_1 + \mathbf{X}_2\} = 3\,P\{\mathbf{X}_1 > \mathbf{X}_2 + \mathbf{X}_3\},$$

   where the last equality follows because the three RVs are iid. The only thing in calculating $P\{\mathbf{X}_1 > \mathbf{X}_2 + \mathbf{X}_3\}$ is in finding the correct integration limits. If u_1, u_2 and u_3 denote the dummy variables for $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{X}_3$, then we have $u_3 \in [0, u_1 - u_2],\ u_2 \in [0, u_1]$ and $u_1 \in [0, 2]$. Therefore

$$P\{\mathbf{X}_1 > \mathbf{X}_2 + \mathbf{X}_3\} = \int_{u_1=0}^{2} \int_{u_2=0}^{u_1} \int_{u_3=0}^{u_1-u_2} \frac{1}{2^3}\, du_3\, du_2\, du_1 \ = \ \frac{1}{6}$$

   and from this, we get the desired probability as $3 \times 1/6 = 1/2$.

2. (a) Let $p_i = p\{\mathbf{XYZ} = i\}$ denote the pmf of $\mathbf{XYZ}$. The RV $\mathbf{XYZ}$ takes on 8 values (2 each for each of $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$), 4 of which are distinct — $1, 2, 4$, and 8.

$$p_1 = \frac{1}{8};\ \ p_2 = \frac{3}{8};\ \ p_4 = \frac{3}{8};\ \ p_8 = \frac{1}{8}$$

   (b) Let $p_i = p\{\mathbf{XY} + \mathbf{YZ} + \mathbf{XZ}\}$ denote the pmf of $\mathbf{XY} + \mathbf{YZ} + \mathbf{XZ}$. This random variable takes on 8 values again (why?), 4 of which are again distinct — $3, 5, 8$ and 12.

$$p_3 = \frac{1}{8};\ \ p_5 = \frac{3}{8};\ \ p_8 = \frac{3}{8};\ \ p_{12} = \frac{1}{8}$$

   (c) Let $p_i = p\{\mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2\}$ denote the pmf of $\mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2$. This random variable takes on 8 values (why?), of which again 4 are distinct — $3, 6, 9$ and 12.

$$p_3 = \frac{1}{8};\ \ p_6 = \frac{3}{8};\ \ p_9 = \frac{3}{8};\ \ p_{12} = \frac{1}{8}$$

3. $\mathbf{U}$ is uniform on $(0, 2\pi)$ and $\mathbf{Z}$, independent of $\mathbf{U}$, is exponential with parameter $\lambda = 1$.

(a) Define two new random variables by

$$\mathbf{X} = \sqrt{2\mathbf{Z}} \cos \mathbf{U}\,, \quad \mathbf{Y} = \sqrt{2\mathbf{Z}} \sin \mathbf{U}$$

The system of equations that follows has a unique solution

$$\begin{array}{rcl} a & = & \sqrt{2z}\ \cos u \\ b & = & \sqrt{2z}\ \sin u \end{array} \Rightarrow \begin{array}{rcl} z_1 & = & \dfrac{a^2+b^2}{2} \\ u_1 & = & \tan^{-1}\left(\dfrac{b}{a}\right) \end{array}$$

The ranges of a and b are: $-\infty < a < \infty$, $-\infty < b < \infty$. The Jacobian of the transformation is

$$J(z,u) = \begin{vmatrix} \dfrac{\cos u}{\sqrt{2z}} & -\sqrt{2z}\ \sin u \\ \dfrac{\sin u}{\sqrt{2z}} & \sqrt{2z}\ \cos u \end{vmatrix} = 1 \neq 0, \quad \text{for all } z, u$$

Therefore

$$f_{X,Y}(a,b) = \frac{f_{Z,U}(z_1,u_1)}{|J(z_1,u_1)|} = \frac{1}{2\pi} e^{-\frac{a^2+b^2}{2}}, \qquad -\infty < a < \infty,\ -\infty < b < \infty$$

(b) From the form of $f_{X,Y}(a,b)$ above, we see that it can be written as

$$f_{X,Y}(a,b) = \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}}\right) \cdot \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{b^2}{2}}\right) = f_X(a) \cdot f_Y(b)$$

where $\mathbf{X}$ and $\mathbf{Y}$ are both distributed as $N(0,1)$.

(c) Define the new random variable $\mathbf{R} = \sqrt{\mathbf{X}^2 + \mathbf{Y}^2}$. There are many ways to calculate the pdf of $\mathbf{R}$. The easiest is to realize that $\mathbf{R} = \sqrt{2\mathbf{Z}}$. We know that $\mathbf{Z} \sim \exp(1)$, i.e.,

$$f_Z(z) = \begin{cases} e^{-z}, & z \geq 0 \\ 0, & z < 0 \end{cases}$$

Therefore, with $v = g(z) = \sqrt{2z}$ as our transformation, $g'(z_1) = 1/\sqrt{2z} = 1/v$. Now we get

$$f_R(v) = \frac{f_z(v^2/2)}{1/\sqrt{2z_1}} = \begin{cases} v e^{-\frac{v^2}{2}}, & v \geq 0 \\ 0, & v < 0 \end{cases}$$

which is plotted in Figure 1.

4. The densities of $\mathbf{X}_1$ and $\mathbf{X}_2$ are given by

$$f_{X_1}(u) = \begin{cases} \frac{1}{2}, & -1 \leq u \leq 1 \\ 0, & \text{elsewhere} \end{cases}, \qquad f_{X_2}(u) = \begin{cases} e^{-u}, & u \geq 0 \\ 0, & u < 0 \end{cases}$$

(a) $\mathbf{W} = \mathbf{X}_1 + \mathbf{X}_2$ and $\mathbf{Z} = \mathbf{X}_1 / \mathbf{X}_2$.

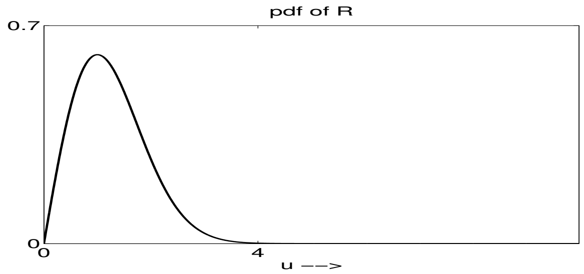


Figure 1: pdf of $\mathbf{Y} = \sqrt{2\mathbf{X}}$ — this is called a Rayleigh density.

pdf of $\mathbf{W} = \mathbf{X}_1 + \mathbf{X}_2$: Since $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent, the pdf of $\mathbf{W}$ is just the convolution of the individual pdfs (here, we are keeping the uniform density fixed and flipping and shifting the exponential density). This pdf is graphed in Figure 3(a).

$$f_W(a) = \begin{cases} 0, & a < -1 \\ \int_{u=-1}^{a} \frac{1}{2} e^{-(a-u)}\, du, & -1 \le a < 1 \\ \int_{u=-1}^{1} \frac{1}{2} e^{-(a-u)}\, du, & a > 1 \end{cases} = \begin{cases} 0, & a < -1 \\ \frac{1}{2}\left(1 - e^{-(a+1)}\right), & -1 \le a < 1 \\ \frac{1}{2}(e - e^{-1})\, e^{-a}\, du, & a > 1 \end{cases}$$

pdf of $\mathbf{Z} = \mathbf{X}_1/\mathbf{X}_2$: Since $\mathbf{X}_2 \ge 0$,

$$F_Z(b) = P\{\mathbf{X}_1/\mathbf{X}_2 \le b\} = P\{\mathbf{X}_1 \le b\mathbf{X}_2\} = \begin{cases} \displaystyle\int_{u=-1}^{0} \int_{v=0}^{u/b} \frac{1}{2} e^{-v} dv\, du, & b < 0 \\ \displaystyle 1 - \int_{u=0}^{1} \int_{v=0}^{u/b} \frac{1}{2} e^{-v} dv\, du, & b \ge 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left(1 + b(1 - e^{1/b})\right), & b < 0 \\ \frac{1}{2}\left(1 + b(1 - e^{-1/b})\right), & b \ge 0 \end{cases}$$

The cases for $b \ge 0$ and $b < 0$ are graphed in Figure 2.

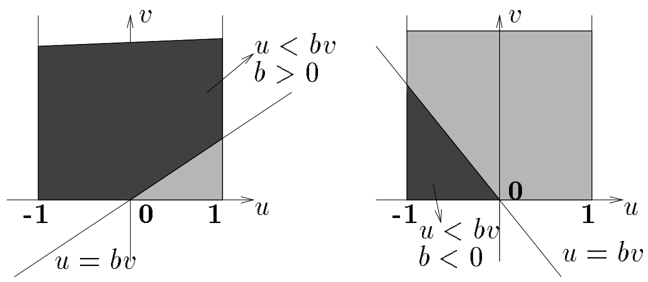
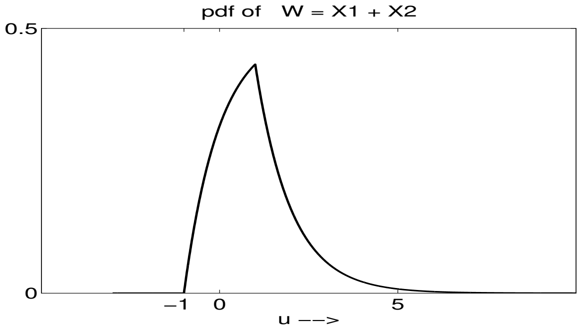


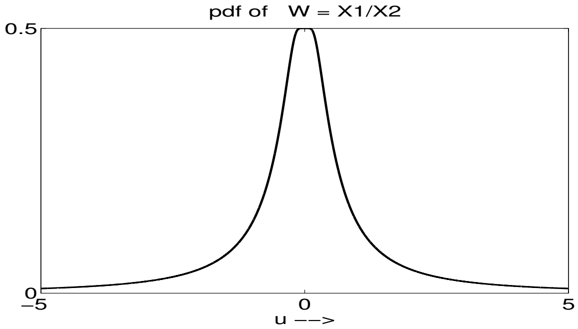
Figure 2: Calculating the pdf for $\mathbf{Z} = \mathbf{X}_1/\mathbf{X}_2$.

From this, the pdf of $\mathbf{Z}$, which is plotted in Figure 3(b), can be obtained by differentiating

$$f_Z(b) = \begin{cases} \frac{1}{2}\left(1 - e^{1/b} + \frac{e^{1/b}}{b}\right), & b < 0 \\ \frac{1}{2}\left(1 - e^{-1/b} - \frac{e^{-1/b}}{b}\right), & b \ge 0 \end{cases}$$



(a)



(b)

Figure 3: pdfs for Problem 4a.

(b) To find the joint pdf $f_{W,Z}(a,b)$, first find the solutions to the set of equations:

$$\begin{array}{rcl} a & = & u+v \\ b & = & u/v \end{array} \quad \Rightarrow \quad \begin{array}{rcl} u_1 & = & \frac{ab}{b+1} \\ v_1 & = & \frac{a}{b+1} \end{array}$$

Now calculate the Jacobian of the transformation at the unique solution point.

$$J(u,v) = \left. \begin{vmatrix} 1 & 1 \\ 1/v & -u/v^2 \end{vmatrix} \right|_{u_1,v_1} = -\frac{1}{v}\left(\frac{u}{v}+1\right)\Bigg|_{u_1,v_1} = \frac{-(b+1)^2}{a}$$

The Jacobian is 0 for $b=-1$, i.e., $\mathbf{X}_1 = -\mathbf{X}_2$, but since $P\{\mathbf{X}_1 = -\mathbf{X}_2\} = 0$ (why?), we don't have to worry about it. Now the joint density is

$$f_{W,Z}(a,b) = \frac{f_{X_1,X_2}(u_1,v_1)}{|J(u_1,v_1)|} = \frac{|a|}{2(b+1)^2}\exp\left(-\frac{a}{b+1}\right)$$

The ranges for a and b have to be evaluated carefully. The uniform density imposes the following limits on a and b: $-1 \le \dfrac{ab}{b+1} \le 1$, and the exponential density imposes $\dfrac{a}{b+1} \ge 0$. These two equations are a little tricky to solve: the second equation yields either $a \ge 0, b \ge -1$ OR $a \le 0, b \le -1$. The first set together with $-1 \le ab/(b+1) \le 1$ yields two regions: $-1 \le a \le 0, -\infty < b \le -1/(1+a)$ and $0 \le a < \infty, -1/(1+a) \le b \le 0$. The second set together with $-1 \le ab/(b+1) \le 1$ yields two regions: $0 \le a \le 1, 0 \le b < \infty$ and $1 \le a < \infty, 0 \le b \le 1/(a-1)$. These are graphed in Figure 4.

5. For the first part $(\mathbf{X},\mathbf{Y})$ is uniformly distributed on the interior of the square with vertices at $(0,0),(0,1),(1,1)$ and $(1,0)$. The marginals in this case are uniform on $(0,1)$, and the joint density is therefore a product of the marginals. (Whenever the joint pdf is constant over a rectangular region, you will be able to write it as a product of the marginals.)

   (a) From the above reasoning, $\mathbf{X}$ and $\mathbf{Y}$ are independent and hence uncorrelated.

   (b) Now we determine if $\mathbf{Z} = \mathbf{X}+\mathbf{Y}$ and $\mathbf{W} = \mathbf{X}-\mathbf{Y}$ are uncorrelated and/or independent. First calculate the covariance of $\mathbf{Z}$ and $\mathbf{W}$ and check to see if it is zero. For starters,

$$E[\mathbf{Z}] = E[\mathbf{X}] + E[\mathbf{Y}] = 1, \quad E[\mathbf{W}] = E[\mathbf{X}] - E[\mathbf{Y}] = 0 \quad \text{(why is this?)}$$
$$\mathrm{Var}(\mathbf{Z}) = \mathrm{Var}(\mathbf{X}) + \mathrm{Var}(\mathbf{Y}) = 1/6, \quad \mathrm{Var}(\mathbf{W}) = \mathrm{Var}(\mathbf{X}) + \mathrm{Var}(\mathbf{Y}) = 1/6$$

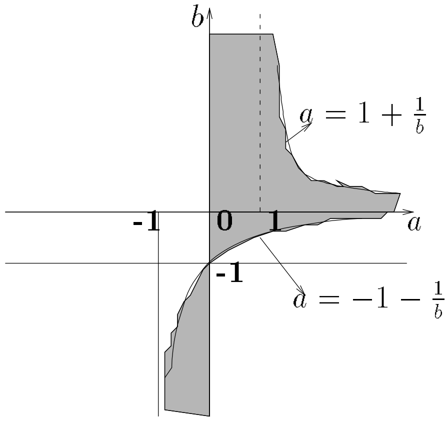


Figure 4: Figure for Problem 4b.

So, $\text{Cov}(\mathbf{Z}, \mathbf{W}) = E[\mathbf{ZW}] - E[\mathbf{Z}]E[\mathbf{W}] = E[(\mathbf{X}+\mathbf{Y})(\mathbf{X}-\mathbf{Y})]$ since $\mathbf{W}$ is zero mean. $E[(\mathbf{X}+\mathbf{Y})(\mathbf{X}-\mathbf{Y})] = E[\mathbf{X}^2 - \mathbf{Y}^2] = E[\mathbf{X}^2] - E[\mathbf{Y}^2] = 0$ because $E[\mathbf{X}^2] = E[\mathbf{Y}^2] = 1/3$; hence they are uncorrelated. To determine if $\mathbf{Z}$ and $\mathbf{W}$ are independent, we have to evaluate their joint density. First solve the equations $a = u + v$ and $b = u - v$, where a and b are particular values for $\mathbf{Z}$ and $\mathbf{W}$ respectively. The unique solution is: $u_1 = \dfrac{a+b}{2}$, $v_1 = \dfrac{a-b}{2}$. The Jacobian of the transformation is

$$J(u, v) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$$

Therefore

$$f_{Z,W}(a, b) = \frac{f_{X,Y}(u, v)}{|J(u_1, v_1)|} = \frac{1}{2}, \quad 0 \leq \frac{a+b}{2} \leq 1, \; 0 \leq \frac{a-b}{2} \leq 1$$

and this is graphed in Figure 5(b) — we see that the support of the joint pdf is a rhombus, and hence $\mathbf{Z}$ and $\mathbf{W}$ cannot be independent.
So $\mathbf{X}+\mathbf{Y}$ and $\mathbf{X}-\mathbf{Y}$ are uncorrelated but not independent.

(c) $(\mathbf{X}, \mathbf{Y})$ uniformly distributed on the interior of a square with corners at $(2, 0)$, $(1, 1)$, $(0, 0)$, and $(1, -1)$.

Clearly $\mathbf{X}$ and $\mathbf{Y}$ are not independent since the support (or "footprint") of the pdf is not rectangular. You can also determine this by calculating the marginals, which work out to be as shown in Figure 5(a). To determine if they are correlated, we calculate the covariance. $E[\mathbf{Y}] = 0$ (**be careful**, $E[\mathbf{X}] \neq 0$), and therefore $E[\mathbf{X}]E[\mathbf{Y}] = 0$.

$$\begin{aligned} \text{Cov}(\mathbf{X}, \mathbf{Y}) &= E[\mathbf{XY}] - E[\mathbf{X}]E[\mathbf{Y}] = E[\mathbf{XY}] \\ &= \int_{u=0}^{1} \int_{v=-u}^{u} \frac{1}{2} uv \, dv \, du + \int_{u=1}^{2} \int_{v=u-2}^{2-u} \frac{1}{2} uv \, dv \, du = 0 \end{aligned}$$

Therefore $\mathbf{X}$ and $\mathbf{Y}$ are uncorrelated. While independent $\Rightarrow$ uncorrelated, the reverse does not necessarily hold (as this problem illustrates), i.e., **uncorrelated does not imply independent**.

(d) Again, check and see if $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ and $\mathbf{W} = \mathbf{X} - \mathbf{Y}$ are uncorrelated and/or independent. In this case, $E[\mathbf{X}] = 1$ and $E[\mathbf{Y}] = 0$. In fact, you can see that the random variables $\mathbf{X}$ and $\mathbf{Y}$ in this case just correspond to $\mathbf{Z}$ and $\mathbf{W}$, respectively, from the **first joint density**. Therefore $\mathrm{Var}(\mathbf{X}) = \mathrm{Var}(\mathbf{Y}) = 1/6$. Two densities with the same shape but different means will have the same variances.

$$E[\mathbf{X}^2] = \mathrm{Var}(\mathbf{X}) + (E[\mathbf{X}])^2 = \frac{1}{6} + 1 = \frac{7}{6}$$
$$E[\mathbf{Y}^2] = \mathrm{Var}(\mathbf{Y}) + (E[\mathbf{Y}])^2 = \frac{1}{6}$$

Therefore,

$$\begin{aligned}\mathrm{Cov}(\mathbf{Z}, \mathbf{W}) &= E[(\mathbf{X}+\mathbf{Y})(\mathbf{X}-\mathbf{Y})] - E[(\mathbf{X}+\mathbf{Y})]E[(\mathbf{X}-\mathbf{Y})] \\ &= E[\mathbf{X}^2 - \mathbf{Y}^2] - (E[\mathbf{X}])^2 = 1 - 1 = 0\end{aligned}$$

and the RVs are uncorrelated. To check for independence we have to find the joint density of $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ and $\mathbf{W} = \mathbf{X} - \mathbf{Y}$ again. This can be obtained in a similar fashion as above to yield

$$\begin{aligned}f_{Z,W}(a,b) = \frac{f_{X,Y}(u,v)}{|J(u_1,v_1)|} &= \frac{1}{4}, \quad 0 \le a \le 2,\ 0 \le b \le 2 \\ &= \frac{1}{2} \cdot \frac{1}{2} = f_Z(a) \cdot f_W(b)\end{aligned}$$

<u>Thus $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ and $\mathbf{W} = \mathbf{X} - \mathbf{Y}$ are independent and hence uncorrelated.</u>

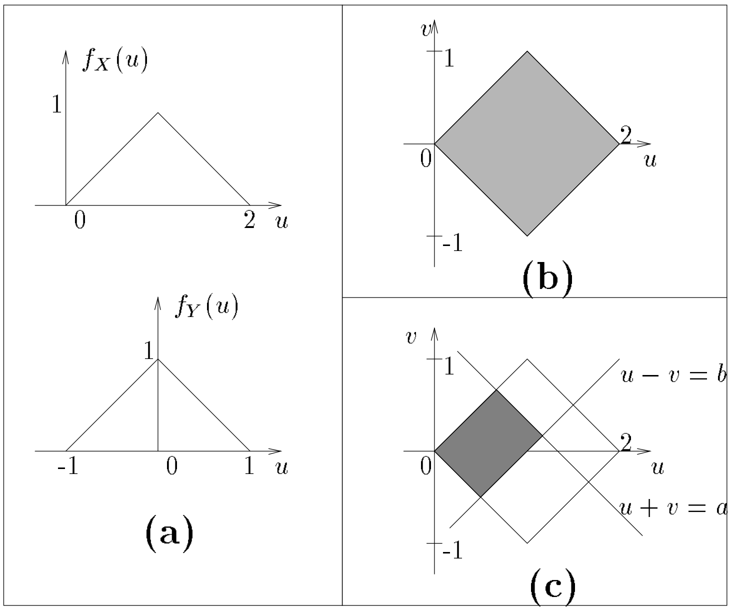


Figure 5: Figure for Problem 5.

In fact, the joint pdf of $\mathbf{X} + \mathbf{Y}$ and $\mathbf{X} - \mathbf{Y}$ has a support that has twice the original area in both cases. (It's also instructive to do this from first principles — find the joint CDF and differentiate it to find the joint pdf; a figure for this is Figure 5(c).)

6. The joint density of $\mathbf{X}$ and $\mathbf{Y}$ is

$$f_{X,Y}(u,v) = c(u^2 - v^2)e^{-u}, \quad 0 \le u < \infty, \ -u \le v \le u$$

The conditional density of $\mathbf{Y}$ given that $\mathbf{X} = u$ is given by $f_{Y|X}(v \,|\, \mathbf{X} = u) = f_{X,Y}(u,v)/f_X(u)$. The interesting thing here is that we do not need to evaluate the constant c, since it cancels out of the numerator and denominator. The marginal density of $\mathbf{X}$ is

$$f_X(u) = \int_{v=-u}^{u} c(u^2 - v^2)e^{-u}\, dv = \frac{4cu^3e^{-u}}{3}, \quad 0 \le u < \infty$$

Therefore,

$$f_{Y|X}(v \,|\, \mathbf{X} = u) = \frac{3c(u^2 - v^2)e^{-u}}{4cu^3e^{-u}} = \frac{3(u^2 - v^2)}{4u^3}, \quad -u \le v \le u$$

Remember here that u is now a **specific** value of the random variable $\mathbf{X}$.

7. $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ are independent random variables having identical density functions $f(u) = e^{-u}$, $0 < u < \infty$. The new RVs are defined by

$$\mathbf{U} = \mathbf{X} + \mathbf{Y}, \qquad \mathbf{V} = \mathbf{X} + \mathbf{Z}, \qquad \mathbf{W} = \mathbf{Y} + \mathbf{Z}$$

If we denote by x, y, and z, the specific values that $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ can take respectively, and by a, b, c, the specific values that $\mathbf{U}, \mathbf{V}, \mathbf{W}$ can take, then we have $a = x + y$, $b = x + z$, $c = y + z$, and $a, b, c \ge 0$ since $x, y, z \ge 0$. The unique solution to this system of equations is

$$x_1 = \frac{a+b-c}{2}, \qquad y_1 = \frac{a-b+c}{2}, \qquad z_1 = \frac{-a+b+c}{2}$$

and the Jacobian of the transformation (note that all the functions involved are linear, and hence continuously differentiable) is

$$J(x,y,z) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2 \ne 0$$

Therefore

$$f_{U,V,W}(a,b,c) = \frac{f_{X,Y,Z}\left(\frac{a+b-c}{2}, \frac{a-b+c}{2}, \frac{-a+b+c}{2}\right)}{|J(x_1,y_1,z_1)|} = \frac{1}{2}\exp\left(-\frac{a+b+c}{2}\right), \qquad a, b, c \ge 0$$

Thus, $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ are also independent RVs.

8. $\mathbf{X}$ and $\mathbf{Y}$ are Gaussian random variables with $E[\mathbf{X}] = 1, E[\mathbf{Y}] = -1, \mathrm{Var}[\mathbf{X}] = 2, \mathrm{Var}[\mathbf{Y}] = 1$ and $\rho_{X,Y} = 0.5$. **(X and Y are not independent.)** From this information, we calculate:

$$E[\mathbf{X}^2] = \mathrm{Var}(\mathbf{X}) + (E[\mathbf{X}])^2 = 3; \quad E[\mathbf{Y}^2] = \mathrm{Var}(\mathbf{Y}) + (E[\mathbf{Y}])^2 = 2; \quad \mathrm{Cov}(\mathbf{X}, \mathbf{Y}) = \frac{1}{\sqrt{2}}$$

(a) $E[2(\mathbf{X}+\mathbf{Y})(\mathbf{X}-\mathbf{Y})] \overset{a}{=} 2E[\mathbf{X}^2 - \mathbf{Y}^2] \overset{b}{=} 2(E[\mathbf{X}^2] - E[\mathbf{Y}^2]) = 2$. In equalities "$a$" and "$b$", we have made use of the linearity of expectation.

(b) $\text{Cov}(\mathbf{U},\mathbf{W}) = E[\mathbf{UW}] - E[\mathbf{U}]\,E[\mathbf{W}]$. Now $E[\mathbf{U}] = E[2\mathbf{X} - 3\mathbf{Y}] = 2E[\mathbf{X}] - 3E[\mathbf{Y}] = 5$, again, by linearity of the expectation. Similarly, $E[\mathbf{W}] = E[2\mathbf{X} + 3\mathbf{Y}] = -1$. Also, $E[\mathbf{UW}] = E[(2\mathbf{X} - 3\mathbf{Y})(2\mathbf{X} + 3\mathbf{Y})] = E[4\mathbf{X}^2 - 9\mathbf{Y}^2] = -6$. Therefore $\text{Cov}(\mathbf{U},\mathbf{W}) = -6 - 5 \cdot (-1) = -1$. We can also calculate the variances of $\mathbf{U}$ and $\mathbf{W}$:

$$\text{Var}(\mathbf{U}) = 4\text{Var}(\mathbf{X}) + 9\text{Var}(\mathbf{Y}) - 2(2)(3)\text{Cov}(\mathbf{X},\mathbf{Y}) = 4(2) + 9(1) - 2(6)(\frac{1}{\sqrt{2}}) = 17 - 6\sqrt{2}$$

$$\text{Var}(\mathbf{W}) = 4\text{Var}(\mathbf{X}) + 9\text{Var}(\mathbf{Y}) + 2(2)(3)\text{Cov}(\mathbf{X},\mathbf{Y}) = 4(2) + 9(1) + 2(6)(\frac{1}{\sqrt{2}}) = 17 + 6\sqrt{2}$$

(c) The trick here is to realize that both $\mathbf{U}$ and $\mathbf{W}$ are Gaussian random variables because they are obtained by performing linear operations on $\mathbf{X}$ and $\mathbf{Y}$, each of which is Gaussian. We have calculated the means of $\mathbf{U}$ and $\mathbf{W}$, their variances and their covariance. All we need to do is calculate the correlation coefficient. $\rho = \text{Cov}(\mathbf{U},\mathbf{W})/\sqrt{\text{Var}(\mathbf{U})\text{Var}(\mathbf{W})} = -1/\sqrt{217}$. Plugging into the general formula below we get

$$f_{U,W}(a,b) = \frac{1}{2\pi\sigma_U\sigma_W\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}\left[\left(\frac{a-\mu_U}{\sigma_U}\right)^2 + \left(\frac{b-\mu_W}{\sigma_W}\right)^2 - 2\rho\left(\frac{a-\mu_U}{\sigma_U}\right)\left(\frac{b-\mu_W}{\sigma_W}\right)\right]}$$

$$f_{U,W}(a,b) = \frac{1}{2\pi \cdot 12\sqrt{6}}\, e^{-\frac{1}{2}\left[\left(\frac{a-5}{17-6\sqrt{2}}\right)^2 + \left(\frac{b+1}{17+6\sqrt{2}}\right)^2 + \frac{2(a-5)(b+1)}{217}\right]}$$

9. **[Extra Credit:]** $\mathbf{A},\mathbf{B},\mathbf{C}$ are independent, and chosen uniformly in the interval $[0,1]$. The roots of the equation $\mathbf{A}x^2 + \mathbf{B}x + \mathbf{C} = 0$ are real if and only if $\mathbf{B}^2 - 4\mathbf{AC} \geq 0$, i.e., $\mathbf{B}^2/4 \geq \mathbf{AC}$. Let us first find the distribution of the random variable $\mathbf{Y} = \mathbf{AC}$. We will then find the probability $P\{\mathbf{B}^2/4 \geq \mathbf{Y}\}$. We see that $\mathbf{Y} \in [0,1]$. Therefore $F_Y(\alpha)$ can be calculated as

$$0 \leq \alpha < 1: \quad P\{\mathbf{Y} > \alpha\} = 1 - F_Y(\alpha) = \int_{u=\alpha}^{1}\int_{v=\alpha/u}^{1} 1 \cdot dv\, du$$
$$= (1 - \alpha + \alpha\ln\alpha)$$

Therefore,

$$F_Y(\alpha) = \begin{cases} 0, & \alpha < 0 \\ \alpha - \alpha\ln\alpha, & 0 \leq \alpha < 1 \\ 1, & \alpha \geq 1 \end{cases}$$

and the probability density is given by

$$f_Y(\alpha) = \begin{cases} 0, & \alpha < 0,\ \alpha \geq 1 \\ -\ln\alpha, & 0 \leq \alpha < 1 \end{cases}$$

Now, since $\mathbf{A},\mathbf{B},\mathbf{C}$ are independent, so are $\mathbf{B}$ and $\mathbf{AC}$ (why?).

$$P\{\mathbf{B}^2/4 \geq \mathbf{Y}\} = -\int_{u=0}^{1}\int_{\alpha=0}^{u^2/4} \ln\alpha\, d\alpha\, du$$
$$= \int_{u=0}^{1}\left(\frac{u^2}{4} + \frac{u^2}{4}\ln\frac{4}{u^2}\right) du,$$

which can be evaluated to yield

$$P\{\mathbf{B}^2/4 \geq \mathbf{Y}\} = \frac{5}{36} + \frac{\ln 2}{6}.$$