University of Illinois

Solutions: Problem Set 12

ECE 313 Fall 1998

1. The joint pmf of $\mathbf{X}$ and $\mathbf{Y}$ is

$$p_{X,Y}(i,j) = \begin{cases} 2^{-(i-1)}\,3^{-j}, & i=1,2,\ldots;\ j=1,2,\ldots \\ 0, & \text{otherwise} \end{cases}$$

   (a) Marginal for $\mathbf{X}$: $p_X(i) = \sum_{j=1}^{\infty} 2^{-(i-1)}\,3^{-j} = 2^{-(i-1)}\frac{1/3}{1-1/3} = 2^{-i} = \frac{1}{2}\,2^{-(i-1)}$ for $i = 1,2,\ldots$.

   Marginal for $\mathbf{Y}$: $p_Y(j) = \sum_{i=1}^{\infty} 2^{-(i-1)}\,3^{-j} = 2\cdot 3^{-j}\frac{1/2}{1-1/2} = 2\cdot 3^{-j} = \frac{2}{3}\,3^{-(j-1)}$, for $j = 1,2,\ldots$.

   Thus we see that $\mathbf{X}$ and $\mathbf{Y}$ are both geometric RVs, with parameters 1/2 and 1/3 respectively. Furthermore, we see that $p_{X,Y}(i,j) = 2^{-(i-1)}\,3^{-j} = (1/2)\,2^{-(i-1)}\cdot(2/3)\,3^{-(j-1)} = p_X(i)\cdot p_Y(j)$. Hence $\mathbf{X}$ and $\mathbf{Y}$ are independent.

   (b) The conditional pmf of $\mathbf{X}$ given that $\mathbf{Y} = k$ is $p_{X|Y}(i\,|\,j=k) = \frac{p_{X,Y}(i,k)}{p_Y(k)}$. Thus,

$$p_{X|Y}(i\,|\,j=k) = \frac{2^{-(i-1)}\,3^{-k}}{2\cdot 3^{-k}} = 2^{-i} = p_X(i), \text{for} i = 1,2,\ldots$$

   (c) Since $p_{X|Y}(i\,|\,j=k) = p_X(i)$, we know that $\mathbf{X}$ and $\mathbf{Y}$ are independent, which we determined in part (a) also.

2. The RVs $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{X}_3$ are independent and identically distributed with pmf $p_{X_i}(0) = 1 - p,\ p_{X_i}(1) = p$.

   (a) $\mathbf{Y} = \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3$. If we let $\mathbf{Z} = \mathbf{X}_1 + \mathbf{X}_2$, then we know from class that $p_Z(u) = p_{X_1}(u) \otimes p_{X_2}(u)$, where '$\otimes$' denotes convolution. Therefore $p_Y(u) = p_Z(u) \otimes p_{X_3}(u) = p_{X_1}(u) \otimes p_{X_2}(u) \otimes p_{X_3}(u)$, i.e., it is the convolution of all three pmfs. The pmfs of $\mathbf{Z}$ and $\mathbf{Y}$ are shown in Figure 1 below.

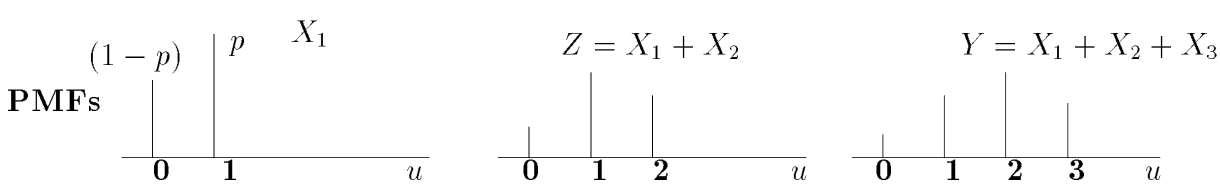


Figure 1: PMFs of $\mathbf{X}_1$, $\mathbf{Z} = \mathbf{X}_1 + \mathbf{X}_2$, $\mathbf{Y} = \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3$.

$$p_Z(0) = (1-p)^2,\ p_Z(1) = 2p(1-p) = \binom{2}{1}p(1-p),\ p_Z(2) = p^2$$

$$p_Y(0) = (1-p)^3,\ p_Z(1) = \binom{3}{1}p(1-p)^2,\ p_Z(2) = \binom{3}{2}p^2(1-p),\ p_Z(3) = p^3$$

   Thus we see that $\mathbf{Z}$ and $\mathbf{Y}$ are binomial RVs with parameters $(2,p)$ and $(3,p)$ respectively.

   (b) If we have $\mathbf{Y} = \mathbf{X}_1 + \cdots + \mathbf{X}_n$, where the $\mathbf{X}_i$'s are independent and identically distributed (iid) Bernoullis(p), then their sum is a binomial RV with parameters (n,p).

(c) If $\mathbf{X} \sim \text{Binomial}(n,p)$ and $\mathbf{Y} \sim \text{Binomial}(m,p)$, then $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ can be equivalently thought of as the sum of $m+n$ independent Bernoullis RVs, each with parameter p. From the above two parts, we see then that the pmf of $\mathbf{Z}$ is binomial with parameter $(m+n,p)$.

3. $\mathbf{X}, \mathbf{Y} \sim$ exponential(λ) and are independent. Therefore $f_X(u) = \lambda e^{-\lambda u}, u \geq 0$ and $f_Y(u) = \lambda e^{-\lambda u}, u \geq 0$.

(a) The random variable $\mathbf{A} = \min\{\mathbf{X}, \mathbf{Y}\}$.

$$\begin{aligned} F_A(u) &= P\{\min\{\mathbf{X},\mathbf{Y}\} \leq u\} = 1 - P\{\min\{\mathbf{X},\mathbf{Y}\} > u\} \\ &\stackrel{a}{=} 1 - P\{\mathbf{X} > u, \mathbf{Y} > u\} \stackrel{b}{=} 1 - P\{\mathbf{X} > u\}\, P\{\mathbf{Y} > u\} \\ &= 1 - (1 - F_X(u))\,(1 - F_Y(u)) \\ &\stackrel{c}{=} 1 - e^{-2\lambda u}, \quad \text{for } u \geq 0 \end{aligned}$$

Equality "a" follows from the fact that if $\min\{X,Y\} > u$ then both X and Y have to be bigger than u. Equality "b" follows from the independence of $\mathbf{X}$ and $\mathbf{Y}$; equality "c" follows because $F_X(u) = F_Y(u) = 1 - e^{-\lambda u}$ for $u \geq 0$. Therefore, the pdf of $\mathbf{A}$ is got by differentiating $F_A(u)$ w.r.t u.

$$f_A(u) = \frac{d}{du}(1 - e^{-2\lambda u}) = \begin{cases} 2\lambda\, e^{-2\lambda u}, & u \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(b) The random variable $\mathbf{B} = \max\{\mathbf{X}, \mathbf{Y}\}$.

$$\begin{aligned} F_B(u) &= P\{\max\{\mathbf{X},\mathbf{Y}\} \leq u\} \stackrel{a}{=} P\{\mathbf{X} \leq u, \mathbf{Y} \leq u\} \\ &\stackrel{b}{=} P\{\mathbf{X} \leq u\}\, P\{\mathbf{Y} \leq u\} = F_X(u) F_Y(u) \\ &\stackrel{c}{=} (1 - e^{-\lambda u})^2, \quad \text{for } u \geq 0 \end{aligned}$$

Equality "a" follows from the fact that if $\max\{X,Y\} \leq u$ then both X and Y have to be smaller than u. Equality "b" follows from the independence of $\mathbf{X}$ and $\mathbf{Y}$; equality "c" follows because $F_X(u) = F_Y(u) = 1 - e^{-\lambda u}$ for $u \geq 0$. Therefore, the pdf of $\mathbf{B}$ is got by differentiating $F_B(u)$ w.r.t u.

$$f_B(u) = \frac{d}{du}(1 - e^{-\lambda u})^2 = \begin{cases} 2\lambda\, e^{-\lambda u}(1 - e^{-\lambda u}), & u \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(c) To evaluate the pdf of $\mathbf{Z} = \mathbf{A} + \mathbf{B}$, we first note that $\mathbf{A}$ and $\mathbf{B}$ are not independent, and so a simple convolution of their individual densities is not enough! To see that they are not independent, if $\mathbf{A} = \mathbf{X}$, then $\mathbf{B} = \mathbf{Y}$, and vice-versa. So we first evaluate their joint CDF $F_{A,B}(u,v)$ in terms of $F_{X,Y}(u,v)$.

Now, $F_{A,B}(u,v) = P\{\mathbf{A} \leq u, \mathbf{B} \leq v\}$. If $u \geq v$ then this probability is just equal to $P\{\mathbf{B} \leq v\}$! This is because if $\max\{\mathbf{X},\mathbf{Y}\} \leq v$, this automatically implies that $\min\{\mathbf{X},\mathbf{Y}\} \leq \max\{\mathbf{X},\mathbf{Y}\} \leq v \leq u$, so $P\{\mathbf{A} \leq u\} = 1$. Therefore $F_{A,B}(u,v) = F_B(v),\ u \geq v$. Now all we need to do is evaluate the joint CDF for the region $u < v$.

For $u < v$, the event $\{\min\{\mathbf{X},\mathbf{Y}\} \leq u, \max\{\mathbf{X},\mathbf{Y}\} \leq v\}$ can occur in three ways as shown in the Figure 3 below

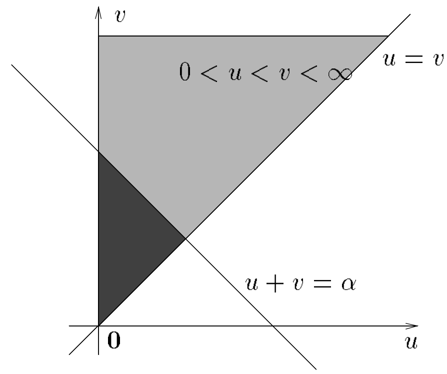


Figure 2: Calculating the pdf of $\mathbf{A}+\mathbf{B}$ using the joint pdf $f_{A,B}(u,v)$.

$$
\begin{aligned}
F_{A,B}(u,v) &= P\{\min\{\mathbf{X},\mathbf{Y}\} \le u, \max\{\mathbf{X},\mathbf{Y}\} \le v\} \\
&\overset{a}{=} P\{\mathbf{X} \le u, \mathbf{Y} \le u\} + P\{\mathbf{X} \le u, u < \mathbf{Y} \le v\} + P\{\mathbf{Y} \le u, u < \mathbf{X} \le v\} \\
&= F_{X,Y}(u,u) + F_{X,Y}(u,v) - F_{X,Y}(u,u) + F_{X,Y}(v,u) - F_{X,Y}(u,u) \\
&\overset{b}{=} F_X(u)\, F_Y(v) + F_X(v)\, F_Y(u) - F_X(u) F_Y(u)
\end{aligned}
$$

Equality "a" follows from the three cases drawn in Figure 3; equality "b" follows from the independence of $\mathbf{X}$ and $\mathbf{Y}$. Therefore, in this particular problem with the same exponential densities for both $\mathbf{X}$ and $\mathbf{Y}$, we have

$$
F_{A,B}(u,v) = \begin{cases} 1 - e^{-2\lambda u} - 2e^{-\lambda v} + 2e^{-\lambda(u+v)}, & 0 \le u < v \\ (1-e^{-\lambda v})^2, & u \ge v \ge 0 \end{cases}
$$

Do you recognize this CDF? You should, since with $\lambda = 1$, this is exactly the same distribution you derived in problem 4(b) in Homework 11! From this, we can obtain the joint pdf of $\mathbf{A}, \mathbf{B}$ as

$$
f_{A,B}(u,v) = \begin{cases} 2\lambda^2\, e^{-\lambda(u+v)}, & 0 \le u < v \\ 0, & u \ge v \ge 0 \end{cases}
$$

We are now ready to calculate the pdf of $\mathbf{Z} = \mathbf{A}+\mathbf{B}$ **(again, you worked this part out in problem 4(f) of Homework 11)**. From Figure 2 below, we see that

$$
\begin{aligned}
F_Z(\alpha) = P\{\mathbf{A}+\mathbf{B} \le \alpha\} &= \int_{u=0}^{\alpha/2} \int_{v=u}^{\alpha-u} 2\lambda^2\, e^{-\lambda u} e^{-\lambda v}\, dv\, du \\
&= 2\lambda^2 \int_{u=0}^{\alpha/2} e^{-\lambda u} \left[-\frac{e^{-\lambda v}}{\lambda} \Bigg|_u^{\alpha-u} \right] du
\end{aligned}
$$

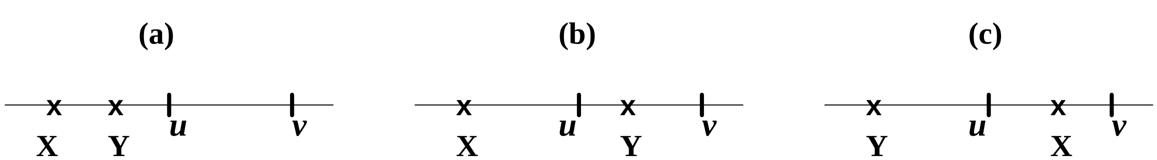


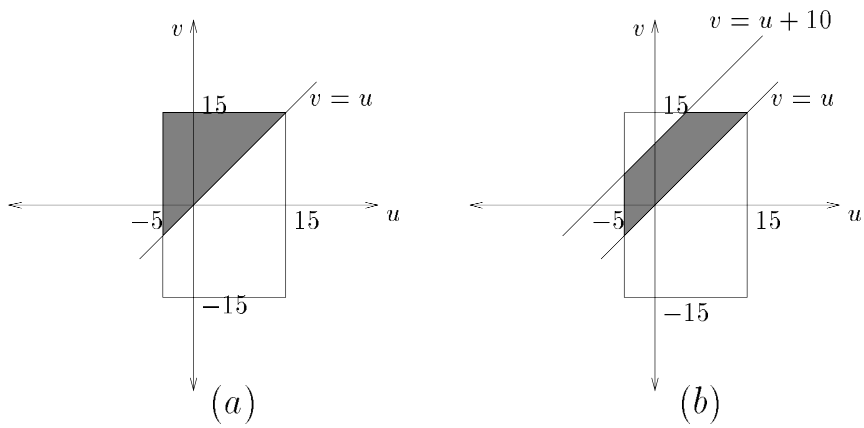
Figure 3: Three ways in which $\min\{\mathbf{X},\mathbf{Y}\} \le u$ and $\max\{\mathbf{X},\mathbf{Y}\} \le v$.

$$= 2\lambda \int_{u=0}^{\alpha/2} (e^{-2\lambda u} - e^{-\lambda\alpha}) du$$
$$= 1 - e^{-\lambda\alpha} - \lambda\alpha e^{-\lambda\alpha}, \qquad \alpha \geq 0$$

Differentiating with respect to α, we get $f_Z(\alpha) = \lambda^2 \alpha e^{-\lambda\alpha}$, $\alpha \geq 0$.

We can easily evaluate the pdf of $\mathbf{X} + \mathbf{Y}$ since $\mathbf{X}$ and $\mathbf{Y}$ are independent, as the convolution of their two densities. We get $f_{X+Y}(\alpha) = \lambda^2 \alpha e^{-\lambda\alpha}$, $\alpha \geq 0$, which is equal to $f_Z(\alpha)$! Is this hard to believe? — it shouldn't be, since $\min\{\mathbf{X}, \mathbf{Y}\} + \max\{\mathbf{X}, \mathbf{Y}\} = \mathbf{X} + \mathbf{Y}$ (verify this), and hence their pdfs have to be equal!

4. Let us denote by $\mathbf{C}$ and $\mathbf{S}$ the random variables corresponding to Calvin's and Susie's times of arrival respectively. If we fix noon to be our reference point, then $\mathbf{C} \sim U[-5, 15]$ and $\mathbf{S} \sim U[-15, 15]$. Since $\mathbf{C}$ and $\mathbf{S}$ are independent RVs, the joint pdf $f_{C,S}(u, v) = \frac{1}{600}$, $-5 \leq u \leq 15, -15 \leq v \leq 15$, and 0 elsewhere, as shown in Figure 4(a) below.



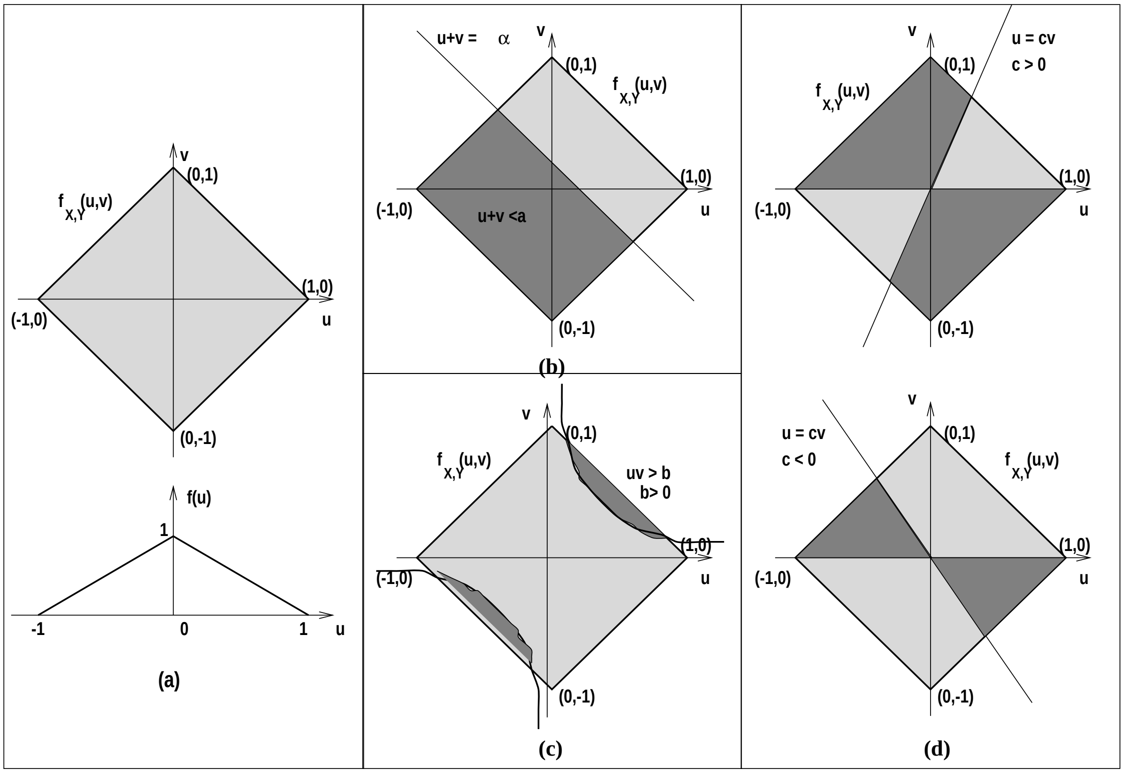
(a) Now the probability that Calvin arrives at "Dead Man's Perch" before Susie does is $P\{\mathbf{C} < \mathbf{S}\}$, which is the area of the triangle in Figure 4(a) multipled by the value of the uniform joint pdf, which is $\frac{1}{600} \times \frac{1}{2} 20 \cdot 20 = \frac{1}{3}$.

(b) Now if Calvin doesn't wait for more than 10 minutes after getting there, the probability that he gets to let loose with his best slushball ever is got from Figure 4(b) as

$$P\{\mathbf{C} < \mathbf{S} < \mathbf{C} + 10\} = \frac{1}{600} \left(\frac{20 \cdot 20}{2} - \frac{10 \cdot 10}{2} \right) = \frac{1}{4}$$

5. The joint pdf of $\mathbf{X}$ and $\mathbf{Y}$ is 1/2 on the rhombus as shown in Figure 5(a).

(a) Right off the bat we can tell that $\mathbf{X}$ and $\mathbf{Y}$ are not independent since the footprint of their joint



pdf is not rectangular. The marginal for $\mathbf{X}$ is given by

$$
\begin{aligned}
f_X(u) &= 0, && u \leq -1, u > 1 \\
&= \int_{v=-1-u}^{1+u} 1/2\, dv \;=\; 1+u, && -1 < u \leq 0 \\
&= \int_{v=u-1}^{1-u} 1/2\, dv \;=\; 1-u, && 0 < u \leq 1
\end{aligned}
$$

This density is graphed in Figure 5(a). Since the joint pdf is symmetric in u and v, it follows that the marginal density for $\mathbf{Y}$ is exactly the same.

(b) $E[\mathbf{X}] = 0$ since the pdf is an even function.
$\mathrm{Var}[\mathbf{X}] = E[\mathbf{X}^2] = \int_{-1}^{0} u^2(1+u)\, du + \int_{0}^{1} u^2(1-u)\, du = \frac{1}{6}$.

(c) The pdf of $\mathbf{A} = \mathbf{X} + \mathbf{Y}$ is best calculated through a diagram.
<u>pdf of $\mathbf{A} = \mathbf{X} + \mathbf{Y}$</u>: Remember that $\mathbf{X}$ and $\mathbf{Y}$ are **not independent** and so you cannot use the convolution formula! But calculating $f_A(a)$ is actually much easier than carrying out a convolution, as can be seen from Figure 5(b):

$$
F_A(a) = P\{\mathbf{A} \leq a\} = \begin{cases} 0\,, & a < -1 \\ \dfrac{1+a}{2}\,, & -1 \leq a < 1 \\ 1\,, & a \geq 1 \end{cases}
$$

and from this we get the pdf of $\mathbf{A}$ to be uniform over the interval $[-1, 1]$, i.e., $f_A(a) = 1/2,\ -1 \le a \le 1$. The CDF and pdf of $\mathbf{A}$ are graphed in Figure 5(a).

(d) $F_C(c) = P\{\mathbf{C} \le c\} = P\{\mathbf{X}/\mathbf{Y} \le c\}$. We have two cases to deal with: $c \ge 0$ and $c < 0$. Furthermore, for each value of c, we have to be careful to take into account two sub-cases: $\mathbf{Y} \ge 0$ and $\mathbf{Y} < 0$. Therefore $P\{\mathbf{X}/\mathbf{Y} \le c\} = P\{\mathbf{X} \le c\mathbf{Y}, \mathbf{Y} \ge 0\} + P\{\mathbf{X} \ge c\mathbf{Y}, \mathbf{Y} < 0\}$.

$\underline{c \ge 0}$: From Figure 5(d) we see that the line $u = cv$ intersects the line $u + v = 1$ at the point $\left(\frac{c}{1+c}, \frac{1}{1+c}\right)$. Therefore

$$P\{\mathbf{X} \le c\mathbf{Y}, \mathbf{Y} \ge 0\} + P\{\mathbf{X} \ge c\mathbf{Y}, \mathbf{Y} < 0\} = 2 \times \left(\frac{1}{4} + \frac{1}{4} \times 1 \times \frac{c}{1+c}\right)$$
$$= \frac{1}{2} + \frac{c}{2(1+c)} = 1 - \frac{1}{2(1+c)}, \qquad 0 \le c < \infty$$

$\underline{c < 0}$: The line $u = cv$ intersects the line $v - u = 1$ at the point $\left(\frac{c}{1-c}, \frac{1}{1-c}\right)$. And so,

$$P\{\mathbf{X} \le c\mathbf{Y}, \mathbf{Y} \ge 0\} + P\{\mathbf{X} \ge c\mathbf{Y}, \mathbf{Y} < 0\} = 2 \times \left(\frac{1}{4} + \frac{1}{4} \times 1 \times \frac{c}{1-c}\right)$$
$$= \frac{1}{2(1-c)}, \qquad -\infty < c < 0$$

Therefore the pdf of $\mathbf{C}$ is given by

$$f_C(c) = \frac{1}{2(1+|c|)^2}, \qquad -\infty < c < \infty$$

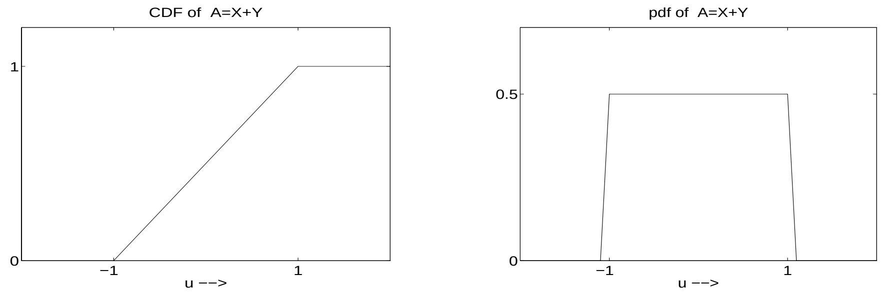
The CDF and pdf of $\mathbf{C}$ are graphed in Figure 5(b).

6. **EXTRA CREDIT PROBLEM:** Three points, $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are chosen at random in the interval $[0, 1]$. This implies that they are all i.i.d (independent and identically distributed) RVs, with CDF $F_X(u) = 0,\ u < 0;\ u,\ 0 \le u < 1$ and $1,\ u \ge 1$.

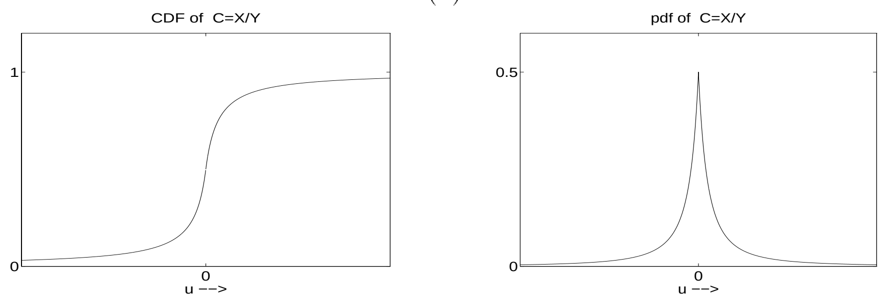
(a) In Problem 3 of this Homework you had to calculate the pdf of the minimum and maximum of two random variables. The same notions apply to three independent random variables also. Let us denote the minimum, middle and maximum of the three RVs by $\mathbf{N}_1, \mathbf{N}_2$ and $\mathbf{N}_3$ respectively. Then the CDFs of $\mathbf{N}_1$ and $\mathbf{N}_3$ are given by

$$F_{N_1}(u) = 1 - (1 - F_X(u))^3 = \begin{cases} 0, & u < 0 \\ 1 - (1-u)^3, & 0 \le u < 1 \\ 1, & u \ge 1 \end{cases}$$
$$F_{N_3}(u) = F_X^3(u) = \begin{cases} 0, & u < 0 \\ u^3, & 0 \le u < 1 \\ 1, & u \ge 1 \end{cases}$$

Now let's find the distribution of the middle point $\mathbf{N}_2$. If $\mathbf{N}_2 \le u$, then at least two of the three



(a)



(b)

points $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ have to be $\leq u$. Therefore

$$\begin{aligned}
F_{N_2}(u) &= P\{\text{at least two of } \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 \leq u\} = 1 - P\{\text{none or exactly one point is} \leq u\} \\
&= 1 - \big(P\{\text{no point is} \leq u\} + P\{\text{exactly one point is} \leq u\}\big) \\
&= 1 - \big((1-F_X(u))^3 + 3F_X(u)(1-F_X(u))^2\big) = 3F_X^2(u)\big(1-F_X(u)\big) + F_X^3(u) \\
&= 0, \quad u < 0; \quad 3u^2 - 2u^3, \quad 0 \leq u < 1; \quad 1, \quad u \geq 1
\end{aligned}$$

Therefore, the pdf of the middle point is

$$f_{N_2}(u) = \begin{cases} 0, & u < 0 \\ 6u - 6u^2, & 0 \leq u < 1 \\ 1, & u \geq 1 \end{cases}$$

(b) Let $\mathbf{Y} = 1 - \mathbf{N}_3$. We need to find the pdf of $\mathbf{Y}$.

$$\begin{aligned}
F_Y(u) &= P\{1 - \mathbf{N}_3 \leq u\} = P\{\mathbf{N}_3 \geq 1 - u\} \\
&= 1 - F_{N_3}(1-u) = 1 - (1-u)^3, \quad 0 \leq u < 1
\end{aligned}$$

But this is just the distribution of the minimum of the three points, $\mathbf{N}_1$! This shouldn't surprise you: if we were to flip the interval around, the maximum would become the minimum and vice-versa. Therefore, the distribution of the distance between $\mathbf{N}_3$ and 1 should be the same as the distribution of the distance between 0 and $\mathbf{N}_1$, which is nothing but the distribution of $\mathbf{N}_1$ itself. Therefore the pdf of $\mathbf{Y}$ is just the pdf of $\mathbf{N}_1$, which is given above.