University of Illinois Spring 1998

ECE 313: Solutions to Problem Set #9

1. **(a)** Let $\mathcal{A}_i$, respectively $\mathcal{B}_i$, denote the event that the person who ranked i-th on the exam is a girl, respectively a boy. Since all the $10!$ rankings are equally likely, we obtain:

$$
\begin{aligned}
p_X(1) = P(X{=}1) &= P(\mathcal{A}_1) &&= \frac{5}{10} = \frac{1}{2} \\
p_X(2) = P(X{=}2) &= P(\mathcal{B}_1\mathcal{A}_2) &&= P(\mathcal{A}_2|\mathcal{B}_1)P(\mathcal{B}_1) = \frac{5}{9}\cdot\frac{5}{10} = \frac{5}{18} \\
p_X(3) = P(X{=}3) &= P(\mathcal{B}_1\mathcal{B}_2\mathcal{A}_3) &&= P(\mathcal{A}_3|\mathcal{B}_2\mathcal{B}_1)P(\mathcal{B}_2|\mathcal{B}_1)P(\mathcal{B}_1) = \frac{5}{8}\cdot\frac{4}{9}\cdot\frac{5}{10} = \frac{5}{36}
\end{aligned}
$$

where we have used the product rule for conditional probabilities. Continuing in this manner, we can further compute:

$$
\begin{aligned}
p_X(4) = P(X=4) &= P(\mathcal{B}_1\mathcal{B}_2\mathcal{B}_3\mathcal{A}_4) &&= \frac{5}{10}\cdot\frac{4}{9}\cdot\frac{3}{8}\cdot\frac{5}{7} = \frac{5}{84} \\
p_X(5) = P(X=5) &= P(\mathcal{B}_1\mathcal{B}_2\mathcal{B}_3\mathcal{B}_4\mathcal{A}_5) &&= \frac{5}{10}\cdot\frac{4}{9}\cdot\frac{3}{8}\cdot\frac{2}{7}\cdot\frac{5}{6} = \frac{5}{252} \\
p_X(6) = P(X=6) &= P(\mathcal{B}_1\mathcal{B}_2\mathcal{B}_3\mathcal{B}_4\mathcal{B}_5) &&= \frac{5}{10}\cdot\frac{4}{9}\cdot\frac{3}{8}\cdot\frac{2}{7}\cdot\frac{1}{6} = \frac{1}{252}
\end{aligned}
$$

It is obvious from the statement of the problem that these are all the possible values of X. Indeed, we can also verify that $p_X(1) + p_X(2) + \cdots + p_X(6) = 1$.

(b) The expected value of X can be easily computed directly by the definition, as the sum of six terms. We have:

$$
E[X] = \sum_{i=1}^{6} i\, p_X(i) = 1\cdot\frac{1}{2} + 2\cdot\frac{5}{18} + 3\cdot\frac{5}{36} + 4\cdot\frac{5}{84} + 5\cdot\frac{5}{252} + 6\cdot\frac{1}{252} = \frac{11}{6}
$$

To find the variance of X, we will also need to compute $E[X^2]$. This can be computed in a similar fashion, as follows:

$$
E[X^2] = \sum_{i=1}^{6} i^2\, p_X(i) = 1\cdot\frac{1}{2} + 4\cdot\frac{5}{18} + 9\cdot\frac{5}{36} + 16\cdot\frac{5}{84} + 25\cdot\frac{5}{252} + 36\cdot\frac{1}{252} = \frac{187}{42}
$$

Thus the variance of X is given by

$$
\mathsf{Var}[X] = E[X^2] - (E[X])^2 = \frac{187}{42} - \frac{121}{36} = \frac{275}{252}
$$

2. Since the balls are placed back in the box after each round, the game consists of independent trials of the same experiment. The probability of success on a single trial, namely the probability of drawing exactly two white balls and two black balls, is given by

$$
p = \frac{\binom{4}{2}\binom{4}{2}}{\binom{8}{4}} = \frac{18}{35}
$$

(a) It is important to realize that the number of rounds X is a geometric random variable, with parameter p computed above. The expected value and variance of a geometric random variable were computed in class and are given by Ross, on page 183. Thus

$$E[X] = \frac{1}{p} = \frac{35}{18} \qquad \text{and} \qquad \mathsf{Var}[X] = \frac{1}{p^2} - \frac{1}{p} = \frac{595}{324}$$

(b) The key observation here is that $Y = e^{-X}$. Thus the easiest way to find the expected value of the reward Y is by using LOTUS as follows:

$$E[Y] = E[e^{-X}] = \sum_{n=1}^{\infty} e^{-n} p_X(n) = \sum_{n=1}^{\infty} e^{-n}(1-p)^{n-1}p$$

where we have used the fact that $p_X(n) = (1-p)^{n-1}p$ for all $n \geq 1$. Indeed, this is the probability mass function of a geometric random variable, as discussed in class. The sum above can be easily computed as follows:

$$E[Y] = \sum_{n=1}^{\infty} e^{-n}(1-p)^{n-1}p = pe^{-1}\sum_{n=1}^{\infty}\big((1-p)e^{-1}\big)^{n-1} = \frac{pe^{-1}}{1-(1-p)e^{-1}}$$

where we used the well-known expression for the sum of a geometric series. Evaluating the above expression for $p = 18/35$, we obtain $E[Y] \simeq 0.230356$.

3. **(a)** The value $c = 3$ can be easily computed using the fact that $\int_{-\infty}^{\infty} f_X(u)\,du = 1$ for any probability density function. Thus

$$1 = \int_{-\infty}^{\infty} f_X(u)\,du = c\int_0^1 (1-u)^2\,du = c\int_0^1 x^2\,dx = \frac{c}{3}$$

(b) We can find both $E[X]$ and $E[X^2]$ by evaluating simple integrals, as follows:

$$E[X] = \int_{-\infty}^{\infty} u\,f_X(u)\,du = 3\int_0^1 u\,(1-u)^2\,du = 0.25$$

$$E[X^2] = \int_{-\infty}^{\infty} u^2\,f_X(u)\,du = 3\int_0^1 u^2\,(1-u)^2\,du = 0.1$$

We can now compute the variance of X follows:

$$\mathsf{Var}[X] = E[X^2] - (E[X])^2 = 0.1 - (0.25)^2 = 0.0375 = \frac{3}{80}$$

(c) The idea in both cases is to express the probability that X satisfies the desired inequality as the probability that X belongs to an interval, or a union of intervals, of the real line. The latter kind of probability can then be computed by integration of the probability density function. Thus

$$P(6X^2 > 5X+1) = P((6X{+}1)(X{-}1) > 0) = P(\{X < -1/6\} \cup \{X > 1\})$$

where we had to solve a quadratic equation to find the factorization of $6X^2 - 5X - 1$. Now it is easy to see that the latter probability is zero, since $f_X(u) = 0$ for $u \notin (0,1)$. Similarly, we solve a quadratic equation to find that

$$P(6X^2 > 7X-2) = P((3X{-}2)(2X{-}1) > 0) = P(\{X < 1/2\} \cup \{X > 2/3\})$$

The latter probability can be computed by integrating the probability density function of X over a single interval, as follows:

$$P(\{X < 1/2\} \cup \{X > 2/3\}) \;=\; 1 \;-\; 3\int_{1/2}^{2/3} (1-u)^2\,du \;=\; \frac{179}{216}$$

4. Notice that in all three parts of this problem, we will make essential use of the LOTUS rule for expected values. In part (**c**), we will also use the linearity of expectation.

(**a**) Here:

$$E[X!] \;=\; \sum_{k=0}^{\infty} k!\,p_X(k) \;=\; \sum_{k=0}^{\infty} k!\,e^{-\lambda}\,\frac{\lambda^k}{k!} \;=\; e^{-\lambda}\sum_{k=0}^{\infty}\lambda^k$$

Thus, using the well-known expression for the sum of a geometric series, we find that

$$E[X!] \;=\; \frac{e^{-\lambda}}{1-\lambda} \qquad \text{for } 0<\lambda<1$$

In our problem, we have $\lambda = 0.5$ and hence $E[X!] = 2\,e^{-0.5} \simeq 1.21306$. Notice that for $\lambda \geq 1$ the geometric sum diverges, and therefore $E[X!] = \infty$.

(**b**) Here:

$$E[X^n] \;=\; \sum_{k=0}^{\infty} k^n\,p_X(k) \;=\; \sum_{k=0}^{\infty} k^n\,e^{-\lambda}\,\frac{\lambda^k}{k!} \;=\; \lambda\sum_{k=1}^{\infty} k^{n-1}\,e^{-\lambda}$$

We will now change the index of summation to $m = k-1$. The limits of the summation over m are 0 and ∞. Thus, we have:

$$E[X^n] \;=\; \lambda\sum_{m=0}^{\infty}(m+1)^{n-1}\,e^{-\lambda}\,\frac{\lambda^m}{m!} \;=\; \sum_{m=0}^{\infty}(m+1)^{n-1}\,p_X(m) \;=\; \lambda\,E[(X+1)^{n-1}]$$

where the second equality follows again from the LOTUS rule. This is precisely what we need to prove, for $\lambda = 0.5$.

(**c**) We will now apply the result of part (**b**) iteratively to find $E[X^3]$. This produces:

$$\begin{aligned} E[X^3] \;&=\; \lambda\,E[(X+1)^2] \;=\; \lambda\,E[X^2+2X+1] = \lambda\big(\lambda E[X+1] + 2E[X] + 1\big) \\ &=\; \lambda\big((\lambda+2)E[X] + \lambda + 1\big) \;=\; \lambda\big((\lambda+2)\lambda + \lambda + 1\big) \;=\; \lambda^3 + 3\lambda^2 + \lambda \end{aligned}$$

where we have used the linearity of expectation, and the fact that $E[X] = \lambda$ for a Poisson random variable X. For $\lambda = 0.5$, the above expression evaluates to $E[X^3] = 1.375$.

5. Let X denote the width of a screw produced by the steel company. Then X is a Gaussian random variable with the following parameters: $X \sim \mathcal{N}(0.9, 9{\cdot}10^{-6})$.

(**a**) This part is asking for the following probability: $P(\{X<0.895\} \cup \{X>0.905\})$. This probability is equal to $2F_X(0.895)$. (Why?!) We can now compute this, as follows:

$$2F_X(0.895) \;=\; 2\Phi\left(\frac{0.895-0.9}{0.003}\right) \;\simeq\; 2\Phi(-1.667) \;=\; 2(1-\Phi(1.667)) \;\simeq\; 0.095$$

(b) Using the result of part (**a**), the maximum allowable value of σ that would achieve the CEO's goal is the largest σ that satisfies the inequality:

$$2\Phi\left(\frac{0.895-0.9}{\sigma}\right) \;=\; 2[1-\Phi(0.005/\sigma)] \;\le\; 0.01$$

Referring to the tables of the standard Gaussian CDF, we find that the above inequality is satisfied for all $\sigma \le 0.0019$.

(c) This part is asking for $P(\{X<0.896\} \cup \{X>0.908\})$. This probability can be computed as follows:

$$1-[F_X(0.908)-F_X(0.896)] \;\simeq\; 1-[\Phi(2.67)-\Phi(-1.33)] = 2-\Phi(2.67)-\Phi(1.33)$$

Referring to the tables of the standard Gaussian CDF, we find that the above is approximately equal to 0.0956. Thus about 9.56% of the screws will be defective.

6. **(a)** Using the general properties of mean and variance, we find that:

$$\begin{aligned} E[2X+5] &= 2E[X]+5 \;=\; 3 \\ \mathsf{Var}[2X+5] &= 2^2\,\mathsf{Var}[X] \;=\; 16 \end{aligned}$$

Notice that there is no need to use the fact that X is Gaussian in this part.

The key idea in the rest of this problem is to define the random variable $Y = \frac{X-\mu}{\sigma} = \frac{X+1}{2}$, and notice that Y is a standard Gaussian random variable. Thus

$$\begin{aligned} P(X<0) &= P\left(\tfrac{X+1}{2} < \tfrac{1}{2}\right) \\ P(-10<X<5) &= P\left(\tfrac{-10+1}{2} < \tfrac{X+1}{2} < \tfrac{5+1}{2}\right) \\ P(|X|>5) &= P(X>5)+P(X<-5) \;=\; P\left(\tfrac{X+1}{2} > \tfrac{5+1}{2}\right) + P\left(\tfrac{X+1}{2} < \tfrac{1-5}{2}\right) \\ P(X^2-3X+2<0) &= P((X-1)(X-2)<0) \;=\; P\left(\tfrac{1+1}{2} < \tfrac{X+1}{2} < \tfrac{2+1}{2}\right) \end{aligned}$$

(Why is the last equality above true?) Now, we can express the probabilities at hand in terms of the random variable Y, as follows:

$$\begin{aligned} P(X<0) &= P(Y<0.5) \\ P(-10<X<5) &= P(-4.5<Y<3) \\ P(|X|>5) &= P(Y>3)+P(Y<-2) \\ P(X^2-3X+2<0) &= P(1<Y<1.5) \end{aligned}$$

The identities $\Phi(-x) = 1-\Phi(x) = Q(x)$ and $Q(-x) = 1-Q(x) = \Phi(x)$ are now used to manipulate answers into the required form, appropriate for Calculators A and B.

$$\begin{aligned} P(Y<0.5) &= \Phi(0.5) &&= 1-Q(0.5) \\ P(-4.5<Y<3) &= \Phi(3)+\Phi(4.5)-1 &&= 1-Q(3)-Q(4.5) \\ P(Y>3)+P(Y<-2) &= 1-\Phi(3)+1-\Phi(2) &&= Q(3)+Q(2) \\ P(1<Y<1.5) &= \Phi(1.5)-\Phi(1) &&= Q(1)-Q(1.5) \end{aligned}$$

These are the required answers, appropriate for calculation using the two Calculators.