University of Illinois Spring 1998

ECE 313: Problem Set #9

Assigned: October 21, 1998
Due: October 28, 1998
Reading: Ross, Chapters 4 and 5
Additional problems: 4.20–4.23, 4.25, 4.27, 4.28, 4.32–4.40, 4.42, 4.43, 4.46, 4.48, 4.50, 4.51

1. Suppose that five boys and five girls are ranked according to their scores on an exam. Assume that no two scores are alike, and that all the $10!$ rankings are equally likely. Let X denote the highest ranking achieved by a girl. For example, if the top-ranked person is a girl, then $X = 1$. If the top-ranked girl is ranked second overall, then $X = 2$, and so forth.

(a) Find the probability mass function of X.

(b) Compute the expected value and variance of X.

2. A box contains 4 white balls and 4 black balls, and the following game is played. Each round of the game consists of randomly selecting four balls from the box. If exactly two of them are white, the game is over. Otherwise, the four balls are placed back in the box, so that it again contains 4 white balls and 4 black balls before the next round is performed. The rounds are repeated until exactly two of the chosen balls are white.

(a) Let X denote the number of rounds in this game. Find the expected value and the variance of X.

(b) When the game is over, we get a reward Y. The value of Y decreases exponentially with the the number of rounds played. Specifically, if the game terminates after n rounds, then $Y = 1/e^n$. Find the expected value of the reward Y.

3. Suppose that the continuous random variable X has probability density function $f_X(u)$ defined as follows: $f_X(u) = c(1-u)^2$ for $0 < u < 1$, and $f_X(u) = 0$ elsewhere.

(a) Find the value of the constant c.

(b) Find the mean and the variance of X.

(c) Compute $P(6X^2 > 5X + 1)$ and $P(6X^2 > 7X - 2)$.

4. Let X denote a Poisson random variable with parameter $\lambda = 0.5$.

(a) Compute $E[X!]$.

(b) Show that $E[X^n] = 0.5\, E[(X+1)^{n-1}]$.

(c) Use this result to compute $E[X^3]$.

5. The width of a screw produced by a steel company is a Gaussian random variable with mean $\mu = 0.9$ cm and standard deviation $\sigma = 0.003$ cm.

(a) If the width specification limits of a customer are 0.9 ± 0.005 cm, and a screw whose width is not within these limits is deemed defective, what percentage of the screws produced by the company will be defective?

(b) The CEO's goal is to make sure that, on the average, no more than 1 in 100 screws produced are defective. What is the maximum allowable value of σ that will enable the company to achieved this goal?

(c) Assume that $\mu = 0.9$ cm and $\sigma = 0.003$ cm as before, but the customer changes specification limits to a minimum of 0.896 cm and a maximum 0.908 cm. What percentage of the screws produced will be defective now?

6. Let X be a Gaussian random variable with mean $\mu = -1$ and variance $\sigma^2 = 4$.

(a) Find the mean and variance of $2X + 5$

Let $\Phi(x)$ denote the CDF of a standard Gaussian random variable, and let $Q(x) = 1 - \Phi(x)$. Suppose that Calculator A can evaluate only $\Phi(x)$ and only for nonnegative values of x. On the other hand, suppose that Calculator B can evaluate only $Q(x)$, again only for $x \geq 0$. Both calculators can perform standard functions, like addition and multiplication. For each of the probabilities in parts (**b**) through (**e**), write down *two* alternative expressions: one for evaluation using Calculator A, and the other for evaluation using Calculator B.

(b) $P(X < 0)$

(c) $P(-10 < X < 5)$

(d) $P(|X| \geq 5)$

(e) $P(X^2 - 3X + 2 < 0)$