University of Illinois

Solutions: Problem Set 8

ECE 313 Fall 1998

1. Let $\mathcal{P}$ denote the event that the student passes the examination. We will use the theorem of total probability to calculate the probability of the student passing with either 3 or 5 examiners. If the student has 3 examiners for his exam, the probability of him passing is given by

$$P(\mathcal{P}) = \left(\frac{1}{3}\right)\sum_{i=2}^{3}\binom{3}{i}(0.8)^i(0.2)^{3-i} + \left(\frac{2}{3}\right)\sum_{i=2}^{3}\binom{3}{i}(0.4)^i(0.6)^{3-i} \ = \ 0.5333$$

If the student has 5 examiners, the probability of him passing is

$$P(\mathcal{P}) = \left(\frac{1}{3}\right)\sum_{i=3}^{5}\binom{5}{i}(0.8)^i(0.2)^{5-i} + \left(\frac{2}{3}\right)\sum_{i=3}^{5}\binom{5}{i}(0.4)^i(0.6)^{5-i} \ = \ 0.5256$$

Therefore, it is marginally advantageous for the student to only have 3 examiners.

2. (a) $\mathbf{X}$, which is a RV denoting the number of passengers that *do not* show up for the flight, is a binomial RV with parameters $(75, 0.2)$. The probability that everyone who shows up for the flight gets a seat can be written in terms of the CDF of $\mathbf{X}$ as

$$P(\texttt{everyone who shows up gets a seat}) = P(\mathbf{X} \geq 5) = 1 - F_X(5^-) = 1 - F_X(4).$$

(b) $\mathbf{Y} \sim$ **Poisson**(15). Therefore, $P(\mathbf{X} \geq 5)$ is easily found as $P(\mathbf{X} \geq 5) = 1 - F_X(4)$ and

$$\begin{aligned}1 - F_X(4) &= 1 - (P\{\mathbf{Y}=0\} + P\{\mathbf{Y}=1\} + P\{\mathbf{Y}=2\} + P\{\mathbf{Y}=3\} + P\{\mathbf{Y}=4\}) \\ &= 1 - e^{-15}\left(\frac{15^0}{0!} + \frac{15^1}{1!} + \frac{15^2}{2!} + \frac{15^3}{3!} + \frac{15^4}{4!}\right) \\ &= 0.999143\end{aligned}$$

The actual probability (from the solutions to HW#5) was found to be 0.999677, and therefore, the relative error is $\dfrac{0.999677 - 0.999143}{0.999677} = 0.05\%$.

(c) Let's compare the binomial probabilities and the Poisson approximations below:

i. $P\{X = 2\}, \ \ n = 8, p = 0.1$:
Binomial probability: $P\{X = 2\} = \binom{8}{2}(0.1)^2(0.9)^6 = 0.1488$.
Poisson approximation: $\lambda = 8 \times 0.1 = 0.8$, and therefore, $P\{X = 2\} = e^{-0.8}\,\frac{(0.8)^2}{2!} = 0.1438$.

ii. $P\{X = 9\}, \ \ n = 10, p = 0.95$:
Binomial probability: $P\{X = 9\} = \binom{10}{9}(0.95)^9(0.05) = 0.3151$.
Poisson approximation: You have to be careful when applying the Poisson approximation here. Notice that the above binomial probability is also equal to $P\{X = 1\}$ with the same n and $p = 0.05$. Now with this scenario, $\lambda = 10 \times 0.05 = 0.5$, and therefore, $P\{X = 9\} = e^{-0.5}\,\frac{(0.5)^1}{1!} = 0.3032$, and the agreement is close. If you had applied the Poisson approximation blindly, you would have obtained $P\{X = 9\} = 0.13$, which is quite inaccurate.

iii. $P\{X=0\}, \ \ n=10, p=0.1$:
Binomial probability: $P\{X=0\}=(0.9)^{10}=0.3487$.
Poisson approximation: $\lambda=10\times 0.1=1$, and so $P\{X=0\}=e^{-1}\frac{1^0}{1!}=0.3679$.

iv. $P\{X=4\}, \ \ n=9, p=0.2$:
Binomial probability: $P\{X=4\}=\binom{9}{4}(0.2)^4(0.8)^5=0.066$.
Poisson approximation: $\lambda=9\times 0.2=1.8$, and so $P\{X=4\}=e^{-1.8}\frac{(1.8)^4}{4!}=0.072$.

You see how the Poisson approximation varies in accuracy depending on how large n is and how small p is.

3. The intensity of the Poisson RV, $\mu=12$ calls/minute.

(a) The parameter $\lambda=\mu t=12\cdot 5/60=1$. If the event that either no call or at least 2 calls arrive at the switchboard is denoted by E, then

$$P(E)=1-P(\texttt{one call at the switchboard})=1-e^{-1}\cdot\frac{1}{1!}=0.6321$$

(b) We are interested in 8 or more occurrences of the event E in 12 independent trials. If we denote this new event by F and set $p=P(E)=0.6321$, then

$$P(F)=\sum_{i=8}^{12}\binom{12}{i}p^i(1-p)^{12-i}=0.5311.$$

The probability is close to one half giving it even chances of occurring, and therefore you shouldn't be overly surprised if this event occurs. However, if you observed that *10 or more intervals* had event E occurring in them, the probability of this event would now be $P(F)=0.1235$, and this might surprise you somewhat. Alternatively, if the rate of calls coming into the switchboard was $\mu=36$ calls/minute (more frequent occurrences of calls) resulting in $P(E)=0.856$, the probability of observing at least 8 intervals with event E occurring in them would jump up to $P(F)=0.9765$, i.e., a very likely event.

(c) The switchboard is to be designed to not drop more than 5% of incoming calls. If the capacity of the switchboard (the number of calls it can service in a minute) is C, then if more than C calls arrive in a minute, they are ignored or "dropped". With the RV X denoting the number of incoming calls per minute, we want to find the minimum value of C such that

$$P\{X>C\}\leq 0.05 \ \text{ or equivalently } \ P\{X\leq C\}>0.95$$

The evaluation of C has to be done numerically, and is best done via a mathematical package on the computer, such as Matlab, Mathematica, or some other. We want to find the minimum value of C such that $\displaystyle\sum_{k=0}^{C}e^{-1}\frac{1}{k!}>0.95$. This turns out to be $C=18$.

4. We will use the Poisson pmf in this problem. Let X be a Poisson distributed random variable that denotes the number of soldiers with the disease. Then $n = 500,\ p = 0.001 \Rightarrow \lambda = 500 \times 0.001 = 0.5$.

   (a) If $\mathcal{E}$ is the event that at least one person has the disease,

   $$P(\mathcal{E}) = 1 - P(\mathcal{E}^c) = 1 - P\{X = 0\} = 1 - e^{-0.5} = 1 - 0.6065 = 0.3935$$

   (b) Let $\mathcal{F}$ be the event that more than one person has the disease. So we need to find $P(\mathcal{F}|\mathcal{E})$. The intersection $\mathcal{F} \cap \mathcal{E} = \mathcal{F}$.

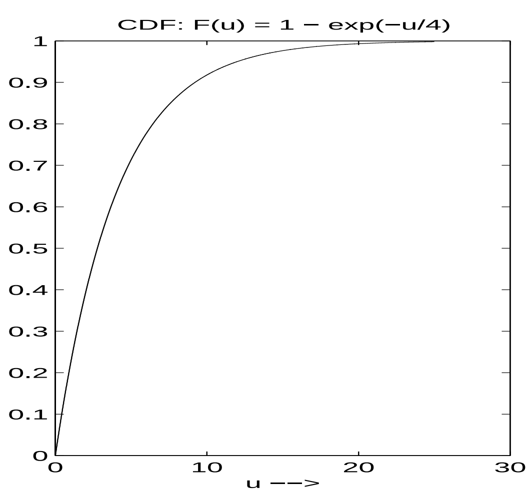
   $$P(\mathcal{F}|\mathcal{E}) = \frac{P(\mathcal{F})}{P(\mathcal{E})} = \frac{\sum_{k>1} e^{-0.5}(0.5)^k/k!}{0.3935} = \frac{0.3935 - P\{X = 1\}}{0.3935} = 0.229$$

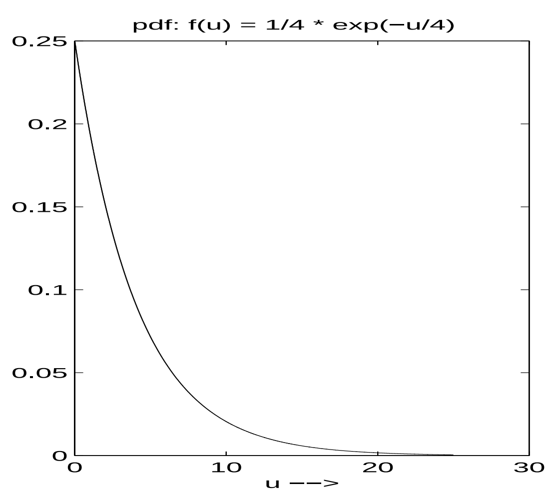
   (c) Since Jones knows he has the disease, he will estimate the probability of more than one person having the disease as the probability that at least one person in 499 has the disease. If we now let Y be a Poisson random variable that counts the number of people out of 499 that have the disease, $P\{Y \geq 1\} = 1 - P\{Y = 0\}$, where λ now equals $499 \times 0.001 = 0.499$. Therefore, $P\{Y \geq 1\} = 1 - e^{-0.499} = 0.3928$.

   (d) Of the first i people tested, one person has the disease. There are $500 - i$ soldiers left, and we want the probability that at least one of these soldiers has the disease. If Y_i is a random variable that denotes the number of soldiers out of $500 - i$ that have the disease, then we want $P\{Y_i \geq 1\} = 1 - P\{Y_i = 0\}$. And in this case, $\lambda = \frac{500-i}{1000}$. Therefore,

   $$P\{Y_i \geq 1\} = 1 - e^{-\frac{500-i}{1000}}$$

5. (a) Since we must have $\int_{-\infty}^{\infty} f_X(u)du = \int_0^{\infty} Ae^{-u/4}du = 1$, we get $A = 1/4$. The corresponding CDF is given by $F_X(u) = 1 - e^{-u/4}$ for $u \geq 0$, and 0 for $u < 0$. The CDF and pdf are graphed below.





   (b) These probabilities can be easily evaluated from the CDF of X. (i) $P(X > 10) = 1 - F_X(10) = e^{-10/4} = 0.082$. (ii) $P(X < 4) = F_X(4) = 1 - e^{-4/4} = 0.6321$.

(c) Just as above, $P[E(b)] = e^{-b/4}$. Now, $P[E(a+b) \mid E(a)] = P[E(a+b) \cap E(a)]/P[E(a)] = P[E(a+b)]/P[E(a)]$ (why?). $P[E(a+b)] = e^{-(a+b)/4}$, and therefore $P[E(a+b) \mid E(a)] = e^{-(a+b)/4}/e^{-a/4} = e^{-b/4} = P[E(b)]$. Thus we see that the probability of being served for more than b minutes **after** being served for (a possibly unknown) a number of minutes, is the same as the probability of being served for more than b minutes starting from "scratch". In other words, it doesn't matter that a customer has already been served for a while — loosely speaking, the **remaining** service time is independent of the **elapsed** service time. It is as if the server (in this case, a bank teller) has no memory of how long he/she/it has already served the customer — this property of the exponential distribution is called the **memoryless** property.

(d) From the memoryless property proved above, the probability that you will be served within the next 10 minutes is just given by $P(X < 10) = 1 - 0.082 = 0.918$.

(e) **Extra credit problem:** See end for solution.

6. In the discrete domain, the random variable that exhibits the same memoryless property is the geometric RV, say X, whose pmf is given by $p_X(k) = p(1-p)^{k-1}$. The CDF of X is $F_X(k) = 1-(1-p)^k$. Therefore, in the setting of Problem 5, $P[E(b)] = (1-p)^b$ and $P[E(a+b) \mid E(a)] = (1-p)^{a+b}/(1-p)^a = (1-p)^b = P[E(b)]$, where it is understood that a, b are positive integers.

4(e) **Extra credit problem:** This problem makes use of the memoryless property of the exponential random variable. There are two ways of solving this problem — the hard way and the easy way. Let's look at the hard way first.

Hard way: Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ denote the service times of Calvin, Hobbes, and you respectively. These are independent and identically distributed random variables. Let $\mathbf{Min} = \min(\mathbf{X}_1, \mathbf{X}_2)$, and $\mathbf{Max} = \max(\mathbf{X}_1, \mathbf{X}_2)$, i.e., $\mathbf{Min}$ represents the minimum of Calvin's and Hobbes' service times (this determines when the first person finishes getting served) and $\mathbf{Max}$ represents the maximum of their service times (this determines when the later of the two finishes getting served). Let $\mathbf{Z} = \mathbf{Min} + \mathbf{X}_3$, which is the random variable representing the time it takes you to finish being served. The probability we are looking for is $P\{\mathbf{Z} < \mathbf{Max}\}$.

Let us first find the distributions of the RVs $\mathbf{Min}$ and $\mathbf{Max}$. $F_{Min}(u) = P\{\mathbf{Min} \le u\} = 1 - P\{\min(\mathbf{X}_1, \mathbf{X}_2) > u\}$. Now $\min(\mathbf{X}_1, \mathbf{X}_2) > u \Rightarrow \mathbf{X}_1 > u,$ and $\mathbf{X}_2 > u$. Therefore

$$F_{Min}(u) = 1 - P\{\mathbf{X}_1 > u, \mathbf{X}_2 > u\} \overset{(a)}{=} 1 - P\{\mathbf{X}_1 > u\} \cdot P\{\mathbf{X}_2 > u\} \overset{(b)}{=} 1 - \Big(1 - F_X(u)\Big)^2,$$

where equality (a) follows from the independence of $\mathbf{X}_1$ and $\mathbf{X}_2$, and equality (b) follows because they are identically distributed. Here $\mathbf{X}$ represents the exponential RV in the problem, with CDF $F_X(u)$ and pdf $f_X(u)$. Differentiating this to get the density function for $\mathbf{Min}$, we get $f_{Min}(u) = 2(1 - F_X(u))f_X(u)$. Next,

$$\begin{aligned} F_{Max}(u) &= P\{\mathbf{Max} \le u\} = P\{\max(\mathbf{X}_1, \mathbf{X}_2) \le u\} \\ &\overset{(a)}{=} P\{\mathbf{X}_1 \le u, \mathbf{X}_2 \le u\} \overset{(b)}{=} P\{\mathbf{X}_1 \le u\} \cdot P\{\mathbf{X}_2 \le u\} \overset{(c)}{=} F_X^2(u), \end{aligned}$$

where equalities (a), (b), and (c) follow from the definition of the maximum, independence of $\mathbf{X}_1$ and $\mathbf{X}_2$, and their identical distribution respectively. Differentiating this we get the pdf for $\mathbf{Max}$ as $f_{Max}(u) = 2F_X(u) \cdot f_X(u)$.

Now we compute the pdf of $\mathbf{Z} = \mathbf{Min} + \mathbf{X}_3$. This is done through conditional densities and Bayes' rule in the continuous case. First compute the CDF of $\mathbf{Z}$.

$$\begin{aligned} F_Z(u) &= P\{\mathbf{Z} \leq u\} = P\{\mathbf{Min} + \mathbf{X}_3 \leq u\} \\ &\stackrel{(a)}{=} \int_0^u P\{\mathbf{X}_3 \leq u - m \,|\, \mathbf{Min} = m\} f_{Min}(m) dm \\ &= \int_0^u 2F_X(u-m) f_{Min}(m) dm = \int_0^u 2F_X(u-m)\Big(1 - F_X(m)\Big) f_X(m) dm \\ &= \int_0^u 2\lambda(1 - e^{-\lambda(u-m)})\, e^{-\lambda m}\; e^{-\lambda m} dm \end{aligned}$$

where equality (a) follows from Bayes' rule applied to continuous RVs. Notice that the upper limit to the integral is u and not ∞ (Why?). If we take $f_X(u) = \lambda e^{-\lambda u}$ and correspondingly, $F_X(u) = 1 - e^{-\lambda u}$ (in the problem, $\lambda = 1/4$, but it is more convenient and instructive to work with a generic λ), then carrying out the integral above gives $F_Z(u) = 1 + e^{-2\lambda u} - 2e^{-\lambda u}$.

And now we are ready to compute the probability we are looking for: $P\{\mathbf{Z} < \mathbf{Max}\}$. Again using Bayes' rule we get

$$\begin{aligned} P\{\mathbf{Z} < \mathbf{Max}\} &= \int_0^\infty P\{\mathbf{Z} < x \,|\, \mathbf{Max} = x\} f_{Max}(x) dx \\ &= \int_0^\infty \Big(1 + e^{-2\lambda x} - 2e^{-\lambda x}\Big)\, 2(1 - e^{-\lambda x}) \cdot \lambda e^{-\lambda x} dx \\ &= 2\lambda \int_0^\infty \Big(e^{-\lambda x} - 3e^{-2\lambda x} + 3e^{-3\lambda x} - e^{-4\lambda x}\Big)\, dx \\ &= \frac{1}{2} \longrightarrow \mathbf{Isthisamazingorwhat!!} \end{aligned}$$

Thus the probability of you not being the last to finish getting served is quite large $(1/2)$ inspite of you starting later! This seems like a lot of calculations for such a simple answer, and indeed it is. The memoryless property of the exponential RV was made use of very implicitly in the lines of mathematical skullduggery above. Let's make more explicit use of this property below.

Easy way: From the memoryless property of the exponential RV, it *doesn't matter when* you start being served (remember, you had to make use of this to solve 4(d)?). The time to finish getting served for the person who is still in his queue when you start getting served **does not depend on** how long he has already been served. Therefore you can **completely ignore** the service time of the person who finishes first, and just compare your service time $\mathbf{X}_3$ to the service time of the person left, which can be taken as $\mathbf{X}$ (a generic exponential RV with the same distribution as $\mathbf{X}_3$). Thus, the probability we are looking for is $P\{\mathbf{X}_3 < \mathbf{X}\}$, and this is $1/2$ for *any pair* of identically distributed RVs. To see this, use Bayes' rule

$$P\{\mathbf{X}_3 < \mathbf{X}\} = \int_{-\infty}^\infty P\{\mathbf{X}_3 < x \,|\, \mathbf{X} = x\} f_X(x) dx \stackrel{(a)}{=} \int_{-\infty}^\infty F_X(x) f_X(x) dx$$

Substitute $t = F_X(x)$; the limits now change from $\{-\infty, \infty\}$ to $\{0, 1\}$, and $dt = f_X(x)dx$. Therefore the above integral reduces to $\int_0^1 t\,dt = 1/2$. Does this seem paradoxical that even though you start getting served well into the service time of the remaining person, you still have a probability of one-half of finishing earlier? This is a demonstration of the memoryless property. Neat, eh? You knew it couldn't be that hard.