University of Illinois

Problem Set 8

ECE 313
Fall 1998

Assigned : Friday, October 16, 1998
Due : Wednesday, October 21, 1998
Reading : Ross, Chapter 4
Recommended additional problems : 4.1, 4.3, 4.4, 4.7–4.10, 4.12–4.16, 4.19, 4.39, 4.40, 4.42–4.44, 4.47–4.49, 4.51–4.54, 4.61–4.64, 4.66, 4.69, 4.70, 4.73, 4.76, 4.77

1. Ross, p. 180, Problem 45.

2. (a) In Problem 3 of Homework#5 you calculated the probability that all passengers, out of 75, that showed up for a 70–seat flight got seats. This was done via the binomial theorem. If we now let $\mathbf{X}$ be a random variable denoting the number of passengers that *do not* show up for the flight, what kind of random variable is $\mathbf{X}$, and what are its parameters? Now express the probability that everyone who shows up gets a seat in terms of the CDF or pmf of $\mathbf{X}$.

   (b) We can also calculate the above probability (refer to HW#5 solutions) using a Possion approximation to the above random variable $\mathbf{X}$. Let $\mathbf{Y}$ be a Poisson random variable with parameter $\lambda = 75 \times 0.2 = 15$. Use the Poisson pmf to evaluate the probability that all passengers who show up for the flight get seats. Compare this number to the actual probability and find the relative error.

   (c) Ross, p. 181, Problem 58.
   The binomial random variable with parameters (n, p) is reasonably well approximated by a Poisson random variable with parameter $\lambda = np$, when n becomes very large and p becomes small. That is, under the above conditions, the two pmfs are approximately equal (You can try to prove this yourself or look into Ross, p.154 for a proof).

3. The Poisson random variable can be used to capture the random behavior of many physical phenomenon: for instance, the incidence of telephone calls at a switching station. Suppose that telephone calls coming into a switching station obey a Poisson probability law at a rate of 12 a minute.

   (a) What is the probability that, during a 5-second period, either no call or at least 2 calls arrive at the switchboard?

   (b) If you observed 12 five-second intervals, would you be surprised if 8 or more of these had *either* no call in them *or* at least 2 calls in them?

   (c) Poorly modeled call traffic into a switchboard can lead to calls being "dropped" once in a while. This is obviously something that service providers try to avoid as much as possible to maximize revenue (can you imagine AT&T going out of their way to help you?!). If the switchboard is designed to not drop more than 5% of the incoming calls in a minute, what should it's capacity be (that is, how many calls should it be capable of handling a minute)?

4. Ross, p. 182, Problem 65.

5. The length of time (in minutes) that a customer is served at a counter in Busey Bank (this problem could also apply to packets being served at a router in a communication network) is found to be a random variable with a pdf specified by

$$f_X(u) = \begin{cases} Ae^{-u/4}, & u \geq 0 \\ 0\,, & \text{otherwise} \end{cases}$$

   (a) Find the value of A that makes the above a valid pdf, and graph the resulting pdf and associated CDF.

   (b) What is the probability that a given customer is served for (i) longer than 10 minutes, (ii) less than 4 minutes?

   (c) For any real number b, let $E(b)$ denote the event that a customer is served for more than b minutes. Find $P[E(b)]$. For $a, b > 0$, find $P[E(a+b) \mid E(a)]$, that is, find the conditional probability that a customer is served for **at least** b **more** minutes, given that he/she has already been served for **at least** a minutes. How are the two probabilities you calculated related? (You should come to a very interesting conclusion. This property of the exponential probability distribution is called the *memoryless* property.)

   (d) If you walk into Busey Bank and find one customer being served (assume only one counter, and that you don't know how long this customer has already been served), what is the probability that you will be served within the next 10 minutes?

   (e) **Extra credit [10 pts]:** The next day you and two buddies, Calvin and Hobbes, enter Busey Bank to get some cash, and you find two counters operational. Being the youngest of the trio you let Calvin and Hobbes step up to the two counters first. You can only get served when one of them is finished being served. What is the probability that you are *not the last* to finish being served (do not take into account Calvin's obnoxious behavior which would probably hold things up by half an hour at least)?

6. In the previous problem you proved the memoryless property of the exponential random variable (a continuous RV). Similarly, in the discrete domain there is a random variable that exhibits this property. What is your guess for the pmf of this random variable? Demonstrate the memoryless property in a fashion similar to part (c) in the previous question.
   (**Hint:** This is a familar discrete RV.)