University of Illinois Fall 1998

ECE 313: Solutions to Problem Set #7

1. **(a)** By the properties of a pdf we have

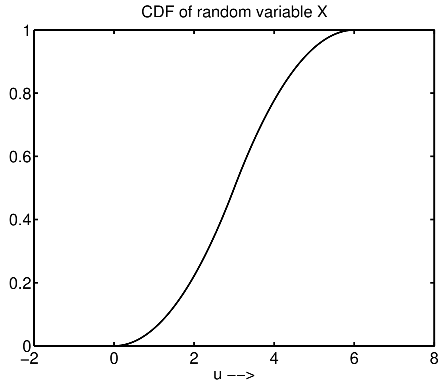
$$1 \;=\; \int_{-\infty}^{+\infty} f_X(u)\,du \;=\; c\int_0^3 u\,du \;+\; c\int_3^6 (6-u)\,du \;=\; 9c$$

Hence $c = 1/9$.

(b) We have a continuous random variable, and therefore $F_X(a) = \int_{-\infty}^{a} f_X(u)\,du$. Thus

$$\begin{aligned}
a < 0: &\quad F_X(a) = 0 \\
0 \le a < 3: &\quad F_X(a) = \frac{1}{9}\int_0^a u\,du = \frac{a^2}{18} \\
3 \le a < 6: &\quad F_X(a) = \frac{1}{9}\int_0^3 u\,du \;+\; \frac{1}{9}\int_3^a (6-u)\,du = \frac{1}{2} - \frac{1}{18}(6-u)^2 \Big|_3^a = 1 - \frac{(6-a)^2}{18} \\
a \ge 6: &\quad F_X(a) = 1
\end{aligned}$$

This function $F_X(u)$ is sketched below.



(c) It is easy to see that $F_X(u)$ is nondecreasing and that $0 \le F_X(u) \le 1$ for all u, either from the sketch above or by analyzing the functions $u^2/18$ and $1-(6-u)^2/18$ in the intervals $[0,3]$ and $[3,6]$. Further, in our case $F_X(-\infty) = F_X(0) = 0$ and $F_X(+\infty) = F_X(6) = 1$. Finally, we see that $F_X(u)$ is continuous everywhere, including the points $u = 0$, $u = 3$, and $u = 6$.

(d) We have $P(A) = P(X > 3) = 1 - F_X(3) = 0.5$. Similarly $P(B) = P(1.5 \le X \le 9) = P(X \le 9) - P(X < 1.5) = F_X(9) - F_X(1.5^-) = 1 - 0.125 = 0.875$.

(e) The intersection of A and B is the event $\{3 < X \le 9\}$ and $P(AB) = P(3 < X \le 9) = P(X > 3) = 0.5 = P(A) \ne P(A)P(B)$. Hence, the two events are *not* independent.

2. Let X denote the chosen number. Then X is a continuous RV, uniformly distributed on the interval $(0,1)$. We have

(a) $P(0.1 \le X < 0.2) \;=\; 0.2 - 0.1 \;=\; 0.1.$

(b) $P(\text{second digit} = 2) \;=\; \sum_{k=0}^{9} P(0.k2 \le X < 0.k3) \;=\; 10 \cdot 0.01 \;=\; 0.1.$

(c) $P(0.3 \le \sqrt{X} < 0.4) \;=\; P(0.09 \le X < 0.16) \;=\; 0.07.$

3. **(a)** $P(X > 5) \;=\; \frac{1}{5}\int_5^{\infty} e^{-u/5}\,du \;=\; -e^{-u/5}\Big|_5^{\infty} \;=\; e^{-1}.$

(b) $P(X < 6) \;=\; \frac{1}{5}\int_0^{6} e^{-u/5}\,du \;=\; -e^{-u/5}\Big|_0^{6} \;=\; 1 - e^{-6/5}.$

(c) $P(5 \leq X \leq 6) = \frac{1}{5}\int_5^6 e^{-u/5}\,du = -e^{-u/5}\Big|_5^6 = e^{-1} - e^{-6/5}$.

(d) $P(X < 6 \mid X > 5) = P(5 < X < 6)/P(X > 5) = (e^{-1} - e^{-6/5})/e^{-1} = 1 - e^{-1/5}$. Note that this is also the unconditional probability that the conversation takes less than one minute.

4. The geometric RV X has a pmf $p_X(k) = (1/2)^k$. X takes on values $k = 1, 2, 3, \ldots$ and therefore, $Y = \sin(\pi X/2)$ takes on values $\{-1, 0, 1\}$ for $k = 4m+3, 2m+2$, and $4m+1$ respectively, $m = 0, 1, 2, \ldots$. Therefore, Y is a discrete RV and its pmf is given by

$$P\{Y = 1\} = p_Y(1) = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{4m+1} = \frac{1/2}{1 - 1/16} = \frac{8}{15}$$

$$P\{Y = 0\} = p_Y(0) = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{2m+2} = \frac{1/4}{1 - 1/4} = \frac{1}{3}$$

$$P\{Y = -1\} = p_Y(-1) = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{4m+3} = \frac{1/8}{1 - 1/16} = \frac{2}{15}$$

In other words, Y takes on the values $-1, 0, 1$ with probabilities $2/15$, $1/3$, and $8/15$ respectively.