University of Illinois Spring 1998

ECE 313: Solutions to Problem Set #6

1. Let us first consider a more general situation. Suppose that event A occurs with probability $p = P(A)$ on a single trial. If A occurs, we will call the trial a success, otherwise we call it a failure. Suppose that the independent trials are repeated indefinitely, and let $\mathcal{A}_{n,m}$ denote the event that n successes occur *before* the occurrence of m failures. Then $\mathcal{A}_{n,m}$ occurs if and only if there are at least n successes among the first $(n+m-1)$ trials. Indeed, if there are at least n successes among the first $(n+m-1)$ trials, then there are at most $m-1$ failures; thus n successes would occur before m failures. On the other hand, if there are at most $n-1$ successes among the first $(n+m-1)$ trials, then there are at least m failures; thus n successes would not occur before m failures. The probability of at least n successes in $(n+m-1)$ trials is given by:

$$P(\mathcal{A}_{n,m}) = \sum_{i=n}^{n+m-1} \binom{n+m-1}{i} p^i (1-p)^{n+m-1-i} \tag{1}$$

as shown in class. We will now apply this result to our problem. In our case, the trials consist of rolling a pair of fair dice. Notice, however, that we *do not care* about those trials where neither 7 nor an even number is rolled. Thus let us call a trial *relevant* if either 7 or an even number result as the outcome. If $\mathcal{S}$ is an event that a seven is rolled on a single trial and $\mathcal{E}$ is an event that an even number is rolled on a single trial, then

$$P(\mathcal{S}) = \frac{1}{6} \qquad \text{and} \qquad P(\mathcal{E}) = \frac{1}{2}$$

Since these events are disjoint, the probability of the event $\mathcal{R}$ that a given trial is relevant is $P(\mathcal{R}) = P(\mathcal{E} \cup \mathcal{S}) = P(\mathcal{E}) + P(\mathcal{S}) = 2/3$. We will call a relevant trial a success if 7 is rolled, otherwise we will call it a failure. The probability of success is then given by

$$p = P(\mathcal{S}|\mathcal{R}) = \frac{P(\mathcal{R}|\mathcal{S})P(\mathcal{S})}{P(\mathcal{R})} = \frac{1 \cdot (1/6)}{(2/3)} = \frac{1}{4}$$

where we have used Bayes inversion formula to find $P(\mathcal{S}|\mathcal{R})$. The answer to the question in our problem can be now obtained from (1) by substituting $p = 0.25$, $n = 2$, and $m = 6$. Thus

$$P(\mathcal{A}_{2,6}) = \sum_{i=2}^{7} \binom{7}{i} (0.25)^i (0.75)^{7-i} = 1 - \sum_{i=0}^{1} \binom{7}{i} (0.25)^i (0.75)^{7-i}$$

is desired the probability of rolling 7 twice before rolling an even number six times. (Exercise: where does the second equality above come from?) Evaluating this expression we find that $P(\mathcal{A}_{2,6}) = 1 - 3^6 \cdot 10/4^7 = 4547/8192 = 0.5550537109375$.

2. The following table is quite helpful in solving this problem:

ω in ...	ABC	ABC^c	AB^cC	AB^cC^c	A^cBC	A^cBC^c	A^cB^cC	$A^cB^cC^c$
value of $X(\omega)$	2	3	0	1	1	2	-1	0
probability	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125

(a) Referring to the table above, the random variable X takes on the values $-1, 0, 1, 2, 3$.

(b) Using the table again, we see that the probability mass function of X is given by $p_X(-1) = p_X(3) = 0.125$, $p_X(0) = p_X(1) = p_X(2) = 0.25$, and $p_X(u) = 0$ for all u not in the set $\{-1, 0, 1, 2, 3\}$. The CDF of X can be easily found from its pmf, as follows

$$F_X(u) = \begin{cases} 0 & u < -1 \\ 0.125 & -1 \le u < 0 \\ 0.375 & 0 \le u < 1 \\ 0.625 & 1 \le u < 2 \\ 0.875 & 2 \le u < 3 \\ 1 & u \ge 3 \end{cases}$$

3. One possible probability measure on $\Omega = \{1, 2, 3, 4, 5\}$ that would satisfy the requirements of parts (**a**) and (**b**) is defined by

$$P(\{1\}) = 0.1\,; \quad P(\{2\}) = 0.1\,; \quad P(\{3\}) = 0.2\,; \quad P(\{4\}) = 0.2\,; \quad P(\{5\}) = 0.4\,;$$

The solution in what follows is with respect to this probability measure.

(**a**) We can now define the random variable X as follows

$$X(i) = i \qquad \text{for } i = 1, 2, 3, 4, 5 \tag{2}$$

It is immediate from the definition that X takes the values $1, 2, 3, 4, 5$ with probabilities $0.1, 0.1, 0.2, 0.2, 0.4$, as desired. Notice that there are other ways to define the random variable X to satisfy this property. For instance, we could define

$$X(1) = 2\,; \qquad X(2) = 1\,; \qquad X(i) = i \qquad \text{for } i = 3, 4, 5 \tag{3}$$

(**b**) The random variable Y could be defined as follows

$$Y(3) = \sqrt{2}\,; \qquad Y(1) = Y(4) = \sqrt{3}\,; \qquad Y(2) = Y(5) = \pi \tag{4}$$

Again, this definition of Y satisfies the required property, but it is not unique. Why?

(**c**) It is easy to see that the random variable $Z = XY$ takes 5 different values, depending on the five possible outcomes in Ω. For X and Y defined in (2) and (4), the possible values of Z are

$$\sqrt{3},\ 2\pi,\ 3\sqrt{2},\ 4\sqrt{3},\ 5\pi$$

taken with probabilities $0.1, 0.1, 0.2, 0.2, 0.4$ respectively. If the random variables X, Y were defined differently, then the answer could be different. For instance, if X is defined as in (3) then Z takes the values $2\sqrt{3}, \pi, 3\sqrt{2}, 4\sqrt{3}, 5\pi$.

4. (**a**) Here $F_X(u)$ is not nondecreasing ($F_X(1) = 1$; $F_X(2) = 0$), and hence not a valid CDF.

(**b**) Here $F_X(u)$ is a valid CDF. It is continuous everywhere except at $u = 0$, where it is right-continuous. We have $P\{|X| > 0.5\} = P\{X < -0.5\} + P\{X > 0.5\}$ and

$$P\{X < -0.5\} + P\{X > 0.5\} = F_X(-0.5^-) + [1 - F_X(0.5)] = 0.5e^{-1} + 0.25e^{-1.5}$$

(**c**) Here $F_X(u)$ is not right-continuous at $u = 0$, and therefore not a valid CDF.

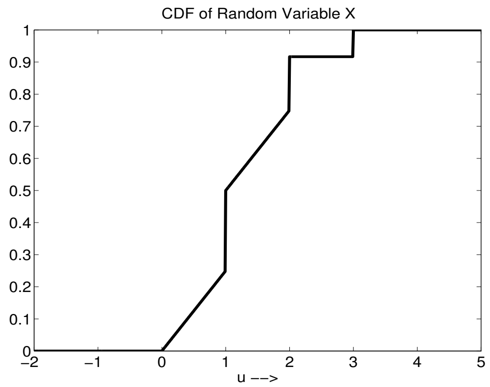

Figure 1: Plot of CDF of the random variable X. Notice how all the properties of a valid CDF are satisfied.

5. (**a**) The probability mass at any point is the height of the discontinuity or step (if one exists) at that point. From Figure 1 we see that

$$P\{X = 1\} = \frac{1}{4}; \qquad P\{X = 2\} = \frac{1}{6}; \qquad P\{X = 3\} = \frac{1}{12}.$$

(b) $P\{\frac{1}{2} < X < \frac{3}{2}\} = F_X(3/2) - F_X(1/2)$ (since $b = 3/2$ is a point of continuity of $F_X(b)$).
Therefore $P\{\frac{1}{2} < X < \frac{3}{2}\} = 5/8 - 1/8 = 1/2$.

6. $P\{X = i\}$ is the probability that number of socks Calvin has to pick out before he gets the first pair is i. This means that the first $i-1$ socks were all from different pairs. The number of ways to choose $(i-1)$ out of the 10 pairs is $\binom{10}{i-1}$, and for each pair chosen, you have one of 2 socks that can be picked. The i^{th} sock has to be a matching sock of one of the first $(i-1)$ socks picked. Of course, $P\{X = i\} = 0$ for $i = 1$ and $i > 11$.
Now the total number of ways of picking i socks out of 20 in this case is **not** $\binom{20}{i}$ as we might think, but actually $\binom{20}{i-1}$ $(20-(i-1))$. This is because order does not matter for the first $(i-1)$ socks only; however the position of the i^{th} sock matters because it cannot occur anywhere else in the sequence (otherwise Calvin might have got a pair earlier). So the first $(i-1)$ socks can be picked in $\binom{20}{i-1}$ ways, and the i^{th} sock can then be picked in $(20-(i-1))$ ways. Therefore

$$P\{X = i\} = \begin{cases} \dfrac{\dbinom{10}{i-1} 2^{i-1}(i-1)}{\dbinom{20}{i-1}(21-i)} & i = 2, 3, \ldots, 11 \\ 0 & i = 1, 12, 13, \ldots, 20 \end{cases} \tag{5}$$

If Calvin now changes his strategy and picks up **two** socks at a time, we let the random variable Y denote the number of **pairs** of socks that he picks before he picks a pair that result in at least one matching pair of socks. There are two ways of doing this. The clever way is to observe that each term of the pmf of Y is the sum of two successive terms in the pmf of X, i.e.,

$$P\{Y = i\} = P\{X = 2i-1\} + P\{X = 2i\} \tag{6}$$

Why is this? Well, think of the two socks in every pair as being picked out sequentially. If $Y = i$, then the first $(i-1)$ pairs, i.e., $2(i-1)$ socks yielded no match; and the i^{th} pair, i.e., either the $(2i-1)^{th}$ or the $(2i)^{th}$ sock resulted in a matching pair (and maybe both of them resulted in matching pairs). Therefore, $P\{Y = i\} = P\{X = 2i-1\} + P\{X = 2i\}$ (convince yourselves that this formula takes care of all cases). As one of our fellow students pointed out, this case is the same as the previous case, except that Calvin checks for a match only *every other pick*, and not every pick.

The second way to do this is by trying to find a closed form expression for $P\{Y = i\}$ directly as we did for $P\{X = i\}$. If $\{Y = i\}$, then the previous $(i-1)$ pairs don't contain a single matching pair; and the i^{th} pair contains at least one match. The first part can be tackled in the same way as the first strategy — except, now the first $2(i-1)$ socks have to belong to different pairs. The number of favorable outcomes for this part is $\binom{10}{2(i-1)} \cdot 2^{2(i-1)}$, and therefore the probability is

$$\frac{\binom{10}{2i-2} \cdot 2^{2(i-1)}}{\binom{20}{2i-2}}$$

For the remaining part — the i^{th} pair yields at least one pair, the best way to calculate this is to find the probability of the complement, i.e., no matching pair. The number of outcomes of no matching pairs after the i^{th} pair is picked is $\binom{10-2(i-1)}{2} 2^2$ (because these two socks have to come from two of the $(10-2(i-1))$ different matching pairs left). The total number of outcomes is $\binom{20-2(i-1)}{2}$. And therefore, the probability of getting at least one pair is

$$1 - \frac{\binom{10-2(i-1)}{2} 2^2}{\binom{20-2(i-1)}{2}} = 1 - \frac{\binom{12-2i}{2} 2^2}{\binom{22-2i}{2}}.$$

The total probability of the desired event, and hence $P\{Y=i\}$ is

$$P\{Y=i\}=\begin{cases}\dfrac{\binom{10}{2i-2}2^{2(i-1)}\left[\binom{22-2i}{2}-\binom{12-2i}{2}2^2\right]}{\binom{20}{2i-2}\binom{22-2i}{2}}, & i=1,2,\ldots,6\\[2ex] 0\,, & \text{otherwise}\end{cases}$$

Those that have the enthusiasm can check and see that the above formula can be obtained by using expressions for $P\{X=2i-1\}$ and $P\{X=2i\}$ from Eq.(5) and plugging them in Eq.(6). The pmfs are tabulated below (the pmf of X is zero after $i=11$, and the pmf of Y is zero after $i=6$):

i	1	2	3	4	5	6
$P\{X=i\}$	0	0.0526	0.1053	0.1486	0.1734	0.1734
$P\{Y=i\}$	0.0526	0.2539	0.3468	0.2553	0.0859	0.0055

i	7	8	9	10	11	$12,\ldots$
$P\{X=i\}$	0.1486	0.1067	0.061	0.0249	0.0055	0

7. **(a)** If the largest number selected out of n balls chosen is k, then $P\{Y=k\}$ is simply the probability that $n-1$ balls were selected from numbers $1,2,\ldots,(k-1)$ and one ball is number k. Clearly $k\geq n$. So

$$P\{Y=k\}=\begin{cases}\dfrac{\binom{k-1}{n-1}\cdot 1}{\binom{N}{n}}, & k=n,\ldots,N\\ 0, & \text{otherwise}\end{cases}$$

(b) When we have selection with replacement we use a slightly different strategy to calculate $P\{Y=k\}$. Note that $P\{Y=k\}=P\{Y\leq k\}-P\{Y\leq k-1\}$. The probability $P\{Y\leq k\}$ is simply the probability that the maximum number picked is less than or equal to k, which is the probability that the first m balls picked were from numbers $1,2,\ldots,k$. Therefore $P\{Y\leq k\}=k^m/n^m$. This gives

$$P\{Y=k\}=P\{Y\leq k\}-P\{Y\leq k-1\}=\frac{k^m-(k-1)^m}{n^m},\quad k=1,\ldots,n$$

8. **[Extra Credit]** This is not a straightforward problem. And there are probably many ways to solve it. Here is one way: Let us denote $P(A)$ and $P(B)$ by a and b respectively. Since A and B are independent, $P(AB)=ab$; $P(A^cB^c)=(1-a)(1-b)$ and $P(A\oplus B)=a(1-b)+b(1-a)$. Let us further denote ab and $a+b$ by P and S respectively. We can rewrite $P(AB)$, $P(A^cB^c)$ and $P(A\oplus B)$ in terms of S and P as:

$$P(AB)=P;\qquad P(A^cB^c)=1-S+P;\qquad P(A\oplus B)=S-2P\,.$$

Now we have to show that at least one of P, $1-S+P$, and $S-2P$ is $\geq 4/9$. Let us assume the contrary and try to find a contradiction. So assume that

$$\begin{matrix}P<4/9\\ 1-S+P<4/9\\ S-2P<4/9\end{matrix}\quad\Longrightarrow\quad\begin{matrix}P<4/9\\ S-P>5/9\\ S-2P<4/9\end{matrix}\tag{7}$$

The above three inequalities together describe a region in the $S-P$ plane bounded by the 3 straight lines: $P=\frac{4}{9}$, $S-P=\frac{5}{9}$, $S-2P=\frac{4}{9}$. This region is a triangle and is shown in Figure 2; it's three vertices can

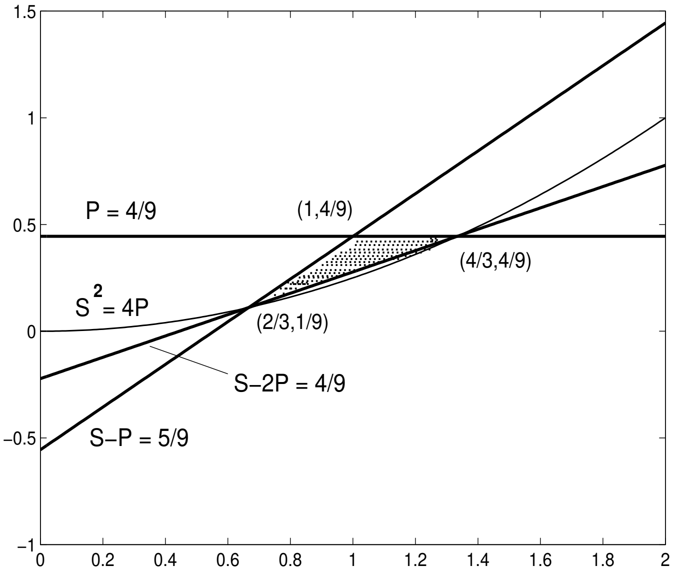

Figure 2: Regions of interest in Problem 8.

easily be found to be $\left(1, \frac{4}{9}\right)$, $\left(\frac{4}{3}, \frac{4}{9}\right)$ and $\left(\frac{2}{3}, \frac{1}{9}\right)$. Thus, any solution to the three inequalities above has to be **in the interior** of this triangle.

However, this is not the end of the story. There is another inequality that S and P have to satisfy and that is $S^2 \geq 4P$. This is because the arithmetic mean of two numbers is always greater than or equal to their geometric mean. Applying this to a and b we get $\frac{a+b}{2} \geq \sqrt{ab} \Rightarrow (a+b)^2 \geq 4ab$, or $S^2 \geq 4P$. The curve $S^2 = 4P$ is also plotted in Figure 2, and the region $S^2 \geq 4P$ is the region **below** the curve. This curve passes through two vertices of the triangle of interest: $(4/3, 4/9)$ and $(2/3, 1/9)$. However, no point in the triangle is in the region described by $S^2 \geq 4P$. Actually you could have found this out without having to plot the curves by using some simple plane geometry.

Therefore there is **no solution** to the inequalities described by Equation (7) and $S^2 \geq 4P$. This is the contradiction we were looking for, and hence at least one of $P(AB)$, $P(A^cB^c)$, $P(A \oplus B)$ is greater than or equal to $4/9$.