University of Illinois Fall 1998

ECE 313: Solutions to Problem Set #5

1. Let the events $B_i,\ i = 1, 2, 3$ denote the events that anchors $B_i,\ i = 1, 2, 3$ anchor the Blobs. Likewise define events S_1 and S_2, and note that $S_2 = S_1^c$. Also, define the events B and S, that the Blobs and Slugs win, respectively. So, $B = S^c$.

(a) To find $P(B_1)$, we will use the Theorem of Total Probability (TTP), and condition B_1 on S_1 and S_1^c.

$P(B_1) = P(B_1 \mid S_1)P(S_1) + P(B_1 \mid S_1^c)P(S_1^c) = (\frac{1}{3})(\frac{1}{2}) + 0(\frac{1}{2}) = 1/6$, because B_1 won't start unless S_1 does and $P(S_1) = 1/2$.

Condition B_2 on B_1 to obtain $P(B_2) = P(B_2 \mid B_1)P(B_1) + P(B_2 \mid B_1^c)P(B_1^c) = 0(\frac{1}{6}) + (\frac{3}{4})(\frac{5}{6}) = 5/8$.

Therefore $P(B_3) = 1 - P(B_1) - P(B_2) = 5/24$.

(b) From the last condition mentioned in the problem, we get $P(B \mid S_2B_2) = 1/2$. Therefore $P(S \mid S_2B_2) = 1 - 1/2 = 1/2$. Similarly, $P(S \mid S_2B_3) = 2/3$.

(c) We need to find the probability $P(S \mid S_2)$. We will use the TTP to condition on B_2, and B_3 (conditioning on B_1 is not necessary since $B_1S_2 = B_1S_1^c = \Phi$).

$$P(S \mid S_2) = P(S \mid S_1^c) = P(S \mid S_1^cB_2)P(B_2 \mid S_1^c) + P(S \mid S_1^cB_3)P(B_3 \mid S_1^c) \tag{1}$$

Note that $P(B_2 \mid S_1^c) = P(B_2 \mid B_1^cS_1^c)P(B_1^c \mid S_1^c)$. This is because the other term in the TTP ($P(B_2 \mid B_1S_1^c)$) doesn't exist, again since $B_1S_1^c = \Phi$. Also, from the conditions given in the problem, event B_2 is independent of S_1^c **given** B_1^c (B_2 is said to be conditionally independent of S_1^c given B_1^c). Hence $P(B_2 \mid B_1^cS_1^c) = P(B_2 \mid B_1^c) = 3/4$. Also $P(B_1^c \mid S_1^c) = 1$. And so, $P(B_2 \mid S_1^c) = 3/4$. Of course, you could have figured this out directly too.
Likewise, $P(B_3 \mid S_2) = P(B_3 \mid B_1^cS_2)P(B_1^c \mid S_2) = \frac{1}{4} \cdot 1 = 1/4$.

Plugging all these values back in Eq. 1, we get

$$P(S \mid S_2) = \frac{1}{2} \times \frac{3}{4} + \frac{2}{3} \times \frac{1}{4} = \frac{13}{24}$$

(d) Using TTP, $P(S) = P(S \mid S_1)P(S_1) + P(S \mid S_2)P(S_2)$. From the previous part, $P(S \mid S_2) = 13/24$. We can calculate $P(S \mid S_1)$ in a similar way.

$$P(S \mid S_1) = P(S \mid S_1B_1)P(B_1 \mid S_1) + P(S \mid S_1B_2)P(B_2 \mid S_1) + P(S \mid S_1B_3)P(B_3 \mid S_1) \tag{2}$$

Note that all three of the Blob anchors can be used now. From the conditions given, $P(S \mid S_1B_1) = \frac{1}{2}$; $P(S \mid S_1B_2) = \frac{2}{3}$; and $P(S \mid S_1B_3) = \frac{3}{4}$.
Now using the TTP,

$$P(B_2) = P(B_2 \mid S_1)P(S_1) + P(B_2 \mid S_2)P(S_2)$$

$$\Rightarrow P(B_2 \mid S_1) = \frac{P(B_2) - P(B_2 \mid S_2)P(S_2)}{P(S_1)} = \frac{\frac{5}{8} - \frac{3}{4} \cdot \frac{1}{2}}{\frac{1}{2}} = \frac{1}{2}$$

In a similar fashion, we can write $P(B_3 \mid S_1) = \dfrac{P(B_3) - P(B_3 \mid S_2)P(S_2)}{P(S_1)} = \dfrac{1}{6}$ (remember that we have calculated all the probabilities on the right hand side of the above equation). Plugging these values back into Eq. 2, we get $P(S \mid S_1) = \frac{1}{2} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{6} = \frac{5}{8}$. Now we can calculate $P(S)$.

$$P(S) = P(S \mid S_1)P(S_1) + P(S \mid S_2)P(S_2) = \frac{5}{8} \cdot \frac{1}{2} + \frac{13}{24} \cdot \frac{1}{2} = \frac{7}{12}.$$

(e) Now we want to find $P(S \mid B_2^c)$. We can again use TTP, but this time, conditioning S on B_2 and B_2^c.

$$P(S) = P(S \mid B_2)P(B_2) + P(S \mid B_2^c)P(B_2^c) \tag{3}$$

Let us denote $P(S \mid B_2^c)$ by p. Again using the TTP:

$$P(S \mid B_2) = P(S \mid S_1 B_2)P(S_1 \mid B_2) + P(S \mid S_2 B_2)P(S_2 \mid B_2) = \frac{2}{3}\cdot\frac{2}{5}+\frac{1}{2}\cdot\frac{3}{5} = \frac{17}{30}$$

We got $P(S_1 \mid B_2) = \frac{2}{5}$ by using Bayes' rule — $P(S_1 \mid B_2) = \dfrac{P(B_2 \mid S_1)P(S_1)}{P(B_2)}$. Also, $P(S_2 \mid B_2) = \dfrac{P(B_2 \mid S_2)P(S_2)}{P(B_2)} = \dfrac{3}{5}$. Plugging back into Eq. 3, we get

$$P(S) = \frac{7}{12} = (\frac{17}{30})(\frac{5}{8}) + p(\frac{3}{8}) \Longrightarrow p = \frac{11}{18}.$$

You can also calculate $P(S \mid B_2^c)$ using the definition: $\dfrac{P(SB_2^c)}{P(B_2^c)} = \dfrac{P(SB_1)+P(SB_3)}{1-P(B_2)}$.

2. What we have here is three independent trials of an experiment with a sample space $\Omega = A \cup B \cup C$ (why?). Furthermore, since A, B, C are disjoint, they form a partition of Ω. In what follows, let E denote the event whose probability we want to compute. We will write E in terms of all the sequences that comply with the conditions of **(a)**, **(b)**, and **(c)**.

(a) Here $E = \{AAA, BBB, CCC\}$. Hence $P(E) = 0.2^3 + 0.5^3 + 0.3^3 = 0.16$.

(b) Here

$$E = \left\{ \begin{array}{cccccc} AAB & AAC & ABA & ACA & BAA & CAA \\ BBA & BBC & BAB & BCB & ABB & CBB \\ CCA & CCB & CAC & CBC & ACC & BCC \end{array} \right\}$$

Hence $P(E) = 3 \cdot \big[\, 0.2^2{\cdot}(0.5+0.3) + 0.5^2{\cdot}(0.2+0.3) + 0.3^2(0.2+0.5)\,\big] = 0.66$.

(c) Here $E = \{ABC, ACB, BAC, BCA, CAB, CBA\}$ and $P(E) = 3! \cdot 0.2{\cdot}0.5{\cdot}0.3 = 0.18$. Note that we could have also found $P(E) = 1 - 0.16 - 0.66$, using parts **(a)** and **(b)**.

(d) Here $E = \{BAA, BAB, BAC, BBA, BBB, BBC, BCA, BCB, BCC\}$. Now define the event $F = \{ABC, ACB, BAC, BCA, CAB, CBA\}$ = all the three vegetables were served during the three days. We have $EF = \{BAC, BCA\}$. Thus

$$P(E \mid F) \;=\; \frac{P(EF)}{P(F)} \;=\; \frac{2{\cdot}P(A)P(B)P(C)}{3!{\cdot}P(A)P(B)P(C)} \;=\; \frac{1}{3}$$

3. Let $\mathcal{A}$ denote the event that all the passengers who show up get seats, and let A_i denote the event that exactly i passengers show up.

(a) In terms of this notation, we have $P(\mathcal{A}) = P\left(\cup_{i=0}^{70} A_i\right) = 1 - P\left(\cup_{i=71}^{75} A_i\right)$. In other words, all we need to find is the probability that 71 or more passengers show up. This probability can be computed straightforwardly as follows:

$$(0.8)^{75} + \binom{75}{74}(0.8)^{74}(0.2) + \binom{75}{73}(0.8)^{73}(0.2)^2 + \binom{75}{72}(0.8)^{72}(0.2)^3 + \binom{75}{71}(0.8)^{71}(0.2)^4$$

Evaluating the above expression produces $P(\mathcal{A}) = 1 - 0.00032331 = 0.999677$.

(b) Let $\mathcal{B}$ be the event that the connecting flight is on time. If $\mathcal{B}^c$ occurs, meaning that the connecting flight is late, then it is obvious that all the passengers who show up get seats. Thus $P(\mathcal{A}|\mathcal{B}^c) = 1$. On the other hand, if the connecting flight is on time, then $\mathcal{A}$ occurs if and only if at most 60 of the other 65 passengers decide to show up for the plane. Thus

$$P(\mathcal{A}|\mathcal{B}) \;=\; 1 - P(\mathcal{A}^c|\mathcal{B}) \;=\; 1 - \sum_{i=61}^{65}\binom{65}{i}(0.8)^i(0.2)^{65-i} \;=\; 0.998255$$

By the theorem of total probability, we have $P(\mathcal{A}) = P(\mathcal{A}|\mathcal{B})P(\mathcal{B}) + P(\mathcal{A}|\mathcal{B}^c)P(\mathcal{B}^c)$. Substituting the above expressions for $P(\mathcal{A}|\mathcal{B})$ and $P(\mathcal{A}|\mathcal{B}^c)$ into this expression, we readily find that $P(\mathcal{A}) = (0.998255)(3/4)+1(1/4) = 0.998691$. Finally, using Bayes inversion rule, we find that the conditional probability that the connecting flight was late, given that all the passengers who showed up got a seat is:

$$P(\mathcal{B}^c|\mathcal{A}) \;=\; \frac{P(\mathcal{A}|\mathcal{B}^c)P(\mathcal{B}^c)}{P(\mathcal{A})} \;=\; \frac{1\cdot(1/4)}{0.998691} \;=\; 0.250328$$

4. Let p_a, p_b, p_c denote, respectively, the probabilities that Alice, Bob, and Carol win when it is *their turn* to roll the dice. Then

$$\begin{aligned} p_a &= P(\text{sum of two fair dice is 8}) &= 5/36 \\ p_b &= P(\text{sum of two fair dice is 9}) &= 4/36 \\ p_c &= P(\text{sum of two fair dice is 10}) &= 3/36 \end{aligned}$$

If Alice wins the game, she either wins on her very first turn, or else Alice, Bob, and Carol are all unsuccessful on their first turns and Alice wins starting from her second turn. Let $\mathcal{X}$ denote the event that Alice, Bob, and Carol are all unsuccessful on their first turns. Then $P(\mathcal{X}) = (1-p_a)(1-p_b)(1-p_c)$. Conditioned on the event $\mathcal{X}$, the win probability P_A for Alice starting from her second turn is *exactly the same* as her win probability at the beginning of the game. Thus P_A is given by

$$P_A \;=\; P(\text{Alice wins on first turn}) + P(\mathcal{X})P_A \;=\; p_a + (1-p_a)(1-p_b)(1-p_c)P_A$$

Since we have already computed the values of p_a, p_b, p_c, we can now easily solve the above equation for the value of P_A. This yields:

$$P_A \;=\; \frac{p_a}{1-(1-p_a)(1-p_b)(1-p_c)} \;=\; \frac{27}{58} \;\simeq\; 0.4656$$

Alternatively, let $\mathcal{A}_i$ be the event that Alice, Bob, and Carol all fail to win the game on their first $i-1$ turns and then Alice wins on her i-th turn. Clearly

$$P(\mathcal{A}_i) \;=\; [P(\mathcal{X})]^{i-1}p_a \;=\; [(1-p_a)(1-p_b)(1-p_c)]^{i-1}p_a \;=\; p_a x^{i-1}$$

where we defined $x \overset{\text{def}}{=} P(\mathcal{X}) = (1-p_a)(1-p_b)(1-p_c)$ for convenience. To find the probability P_A that Alice wins the game, we can sum up the probabilities of all the mutually exclusive events $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots$. This produces the geometric series:

$$P_A \;=\; \sum_{i=1}^{\infty} P(\mathcal{A}_i) \;=\; \sum_{i=1}^{\infty} p_a x^{i-1} \;=\; p_a \sum_{i=0}^{\infty} x^i \;=\; \frac{p_a}{1-x}$$

This is the same result derived by the slicker argument given earlier. Using the same approach, we easily find that:

$$\begin{aligned} P_B &= \frac{(1-p_a)p_b}{1-P(\mathcal{X})} = \frac{93}{290} \simeq 0.3207 \\ P_C &= \frac{(1-p_a)(1-p_b)p_c}{1-P(\mathcal{X})} = \frac{31}{145} \simeq 0.2138 \end{aligned}$$

These expressions follow from the fact that the probability that Bob wins on his first turn is $(1-p_a)p_b$, while the probability that Carol wins on her first turn is $(1-p_a)(1-p_b)p_c$. You can then either condition on the event that Alice, Bob, and Carol all fail on their turns, or use the geometric series approach.

5. **(a)** The idea is to condition on the outcome of the first trial. If A occurs on the first trial, then it will occur an even number of times on n trials if and only if it occurs an odd number of times on the remaining $n-1$ trials. The latter event happens with probability $1-P_{n-1}$. On the other hand, if A does not occur on the first trial, then it will occur an even number of times on n trials if and only if it occurs an even number of times on the remaining $n-1$ trials. This happens with probability P_{n-1}. Thus

$$P_n \;=\; (1-P_{n-1})P(A) + P_{n-1}P(A^c) \;=\; p(1-P_{n-1}) + (1-p)P_{n-1}$$

by the theorem of total probability. This is precisely the expression we need to prove.

(b) Clearly A occurs an even number of times on a single trial if and only if it does not occur on this trial, which happens with probability $1-p$. Computing P_1 using the expression given in the problem, we see that

$$P_1 \;=\; 0.5 + 0.5(1-2p)^1 \;=\; 1-p$$

Thus the expression is correct for $n=1$. We can now establish the rest by induction, using the the result of part (**a**). Thus

$$P_n \;=\; p(1-P_{n-1}) + (1-p)P_{n-1} \;=\; p\Big(0.5 - 0.5(1-2p)^{n-1}\Big) + (1-p)\Big(0.5 + 0.5(1-2p)^{n-1}\Big)$$

where we have used the induction hypothesis to express P_{n-1} as $0.5 + 0.5(1-2p)^{n-1}$. Simplifying the above expression for P_n produces $P_n = 0.5 + 0.5(1-2p)^n$ after some simple manipulations. This establishes the induction step and completes the proof.

6. Suppose that a fair coin is tossed n times. Then there are 2^n possible outcome sequences, each equally likely. Let us call an outcome sequence *acceptable* if it does not contain two consecutive heads, and let F_n denote the number of acceptable sequences. It is obvious that for $n=1$ both sequences H and T are acceptable, so that $F_1 = 2$. For $n=2$ there are three acceptable sequences HT, TH, and TT, so that $F_2 = 3$. In general, we can express F_n in terms of F_{n-1} and F_{n-2} as follows. If the outcome of the first toss is tails, then the sequence of n tosses is acceptable if and only if the remaining $n-1$ tosses produce an acceptable sequence. In other words, the number of acceptable sequences of length n that start with tails is F_{n-1}. One the other hand, if the outcome of the first toss is heads, then the sequence of n tosses is acceptable if and only if the second toss produces tails while the remaining $n-2$ tosses produce an acceptable sequence. Thus the number of acceptable sequences of length n that start with heads is F_{n-2}. Since every acceptable sequence starts with either tails or heads, we obtain:

$$F_n \;=\; F_{n-1} + F_{n-2} \qquad \text{for } n = 3, 4, \ldots$$

The numbers $F_1, F_2, F_3, \ldots$ are known as the *Fibonacci* numbers. They can be easily computed using the recursion above. Here are the first 20 numbers in this sequence:

$$2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711$$

Thus $F_{10} = 144$ and the probability that an acceptable sequence is observed after 10 tosses of a fair coin is $F_{10}/2^{10} = 144/1024 = 9/64 = 0.140625$.

Here is an alternative solution, where instead of counting we work directly with probabilities. Let P_n denote the probability of observing an acceptable sequence after n tosses. Then it is easy to see that $P_1 = 1$, while $P_2 = 3/4$. Conditioning on the outcome of the first toss, and following arguments similar to the above, we obtain the recurrence:

$$P_n \;=\; P(T)P_{n-1} + P(H)P(T)P_{n-2} \;=\; 0.5P_{n-1} + 0.25P_{n-2} \qquad \text{for } n = 3, 4, \ldots$$

Using this recurrence with the initial conditions $P_1 = 1$ and $P_2 = 3/4$, we can easily compute the first few values of P_n, arriving at the same result as before. Notice that this approach works also in the case when the coin is biased, so that the outcome sequences are not equally likely. In general, if the probability of observing heads is p while the probability of observing tails is $q = 1-p$, then $P_n = qP_{n-1} + pqP_{n-2}$.

7. *This problem was moved to Problem Set 6.*

8. [Extra Credit:] Define the event E_i as $\{$ the number i occurs exactly twice in the string $\}$. Then the probability that we want is

$$P(\bigcup_{i=1}^{10} E_i) = \sum_{i=1}^{10} P(E_i) - \sum_{i<j} P(E_iE_j) + \sum_{i<j<k} P(E_iE_jE_k) - \cdots - P(E_1E_2E_3\ldots E_{10})$$

In the above equation, all terms that involve the probabilities of the intersection of more than 3 events are **zero** (why?). Therefore we only have to calculate the first 3 terms in the above sum. Let us examine each one individually.

$$P(E_i) = \binom{6}{2}(0.1)^2(0.9)^4$$

$$P(E_iE_j) = P(E_i)P(E_j \mid E_i) = \binom{6}{2}(0.1)^2\left(\frac{9}{10}\right)^4\binom{4}{2}\left(\frac{1}{9}\right)^2\left(\frac{8}{9}\right)^2 = \binom{6}{2}\binom{4}{2}(0.1)^4(0.8)^2$$

$$P(E_iE_jE_k) = P(E_iE_j)P(E_k \mid E_iE_j) = \binom{6}{2}\binom{4}{2}(0.1)^4(0.8)^2\left(\frac{1}{8}\right)^2 = \binom{6}{2}\binom{4}{2}(0.1)^6$$

Therefore

$$P(\bigcup_{i=1}^{10} E_i) = 10\cdot\binom{6}{2}(0.1)^2(0.9)^4 + \binom{10}{2}\cdot\binom{6}{2}\binom{4}{2}(0.1)^4(0.8)^2 + \binom{10}{3}\cdot\binom{6}{2}\binom{4}{2}(0.1)^6 = 0.73575$$