University of Illinois Fall 1998

ECE 313: Problem Set #5

Assigned: September 21, 1998
Due: September 30, 1998
Reading: Ross, Chapter 3

1. **[20 pts]** Calvin is trying his hand at gambling on a particular game in a tug-of-war tournament on Narg. His friend, a bookie, has given him the following information on the two teams playing: The Slugs and The Blobs.

   Each team must have an anchor, on whose ample girth the teams fortunes depend. The Slugs have only two anchors, S_1 and S_2, of different abilities. The Blobs have three anchors: B_1, their heaviest but tempermental anchor; B_2, their most reliable anchor; B_3, the worst of the three. Furthermore:

   The Blobs will not start B_1 if S_1 does not start. If S_1 does start, the Blobs will start B_1 with probability 1/3.

   The Slugs are equally likely to start S_1 or S_2, **no matter what the Blobs do**.

   B_2 will anchor the Blobs with probability 3/4 if B_1 does not anchor, **no matter what else happens**.

   The probability that the Slugs will win, given S_1 anchors, is $m/(m+1)$, where m is the subscript of the Blob anchor. The probability that the Blobs will win, given S_2 anchors, is $1/m$, where m is again the subscript of the Blob anchor.

   (a) What is the probability that B_m starts, $m = 1, 2, 3$?
   (b) What is the probability that the Slugs win, given that S_2 **and** B_m, $m = 2, 3$ anchored?
   (c) What is the probability that the Slugs win, given that S_2 anchored?
   (d) What is the probability that the Slugs win?
   (e) Given that B_2 does not anchor, what is the probability that the Slugs win?

2. Let A, B, and C denote the events that your mother serves respectively asparagus, broccoli, and cauliflower for dinner. From (bitter?) experience you know that these events are disjoint, and that $P(A) = 0.2$, $P(B) = 0.5$, $P(C) = 0.3$. Your mother makes independent decisions (without taking your opinion into account!) about the vegetable to serve each day. ["No, dear, Cheetos and Oreos are not vegetables ..."]. Over a three day period, what is the probability that:

   (a) she serves the same vegetable on all three days?
   (b) she serves the same vegetable exactly two days out of three?
   (c) she serves different vegetables on the three days?
   (d) she served broccoli on the first day, given that all the three different vegetables were served during the three days?

3. Suppose that 75 passengers hold reservations for a flight from Chicago to Champaign. The airplane has 70 seats only. Each passenger decides independently with probability 0.8 whether to show up for the flight.

(**a**) Find the probability that all passengers who show up get seats.

(**b**) Suppose that 10 passengers are arriving in Chicago on a connecting flight which is late with probability 1/4. If the connecting flight is on time, all 10 show up for the flight to Champaign; otherwise none of the 10 shows up. The remaining 65 passengers decide independently as before (and also independently of the fate of the connecting flight). What is the probability that all passengers who show up get a seat? Given that all the passengers who showed up got a seat, what is the (conditional) probability that the connecting flight was late?

4. Alice, Bob, and Carol play a game in which each player rolls a pair of fair dice in turn, with Alice going first, then Bob, and then Carol, and then Alice again, and so on. When it is Alice's turn to play, the game terminates with a win for Alice if the sum of the two dice she rolls is 8. When it is Bob's turn, the game terminates with a win for Bob if the sum of the two dice he rolls is 9. When it is Carol's turn, the game ends with a win for Carol if the sum of the dice she rolls is 10. The game continues with each player rolling the dice in turn, until one of the three players wins. Find the win probabilities for each player.

5. Suppose that n independent trials of an experiment are performed, and that event A occurs with probability $P(A) = p$ on each trial. Let P_n denote the probability that A occurs an even number of times on n trials.

(**a**) Show that $P_n = p(1 - P_{n-1}) + (1 - p)P_{n-1}$ for $n \geq 2$.

(**b**) Use the result of part (**a**) to prove that $P_n = 0.5 + 0.5(1 - 2p)^n$ for $n \geq 1$.

Notice that if A does not occur at all, then it occurs an even number of times, because zero is an even number. This problem is taken from Ross, theoretical exercise 15, on p. 120.

6. A fair coin is tossed 10 times. What is the probability that two *consecutive* heads do not occur among the 10 outcomes?

7. In successive rolls of a pair of fair dice, what is the probability of rolling "seven" twice (not necessarily on consecutive rolls), before rolling an even number six times (not necessarily on consecutive rolls)? This is taken from Ross, problem 74, on p. 115.

8. **[Extra credit 10 pts]** Suppose that a string of six decimal digits — that is, one of the 10^6 numbers $000000, 000001, 000002, \ldots, 999999$, is drawn independently at random. Find the probability that at least one of the digits $0, 1, \ldots, 9$ appears exactly twice in this string.
Note: No credit given for solutions based on computer programs.