University of Illinois Fall 1998

ECE 313: Solutions to Problem Set #4

1. Let B be the event that the target is hit, and let A be the event that there is a gust of wind. Then, by the theorem of total probability, we have

(a) $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = (0.4)(0.3) + (0.7)(0.7) = 0.61$

Part (**b**) is asking for the conditional probability of the event A^c given that the event B^c occurred. Using Bayes inversion formula, we obtain $P(A^c|B^c) = P(B^c|A^c)P(A^c)/P(B^c)$.

(b) ¿From part (**a**), we have $P(B^c) = 0.39$. Furthermore $P(B^c|A^c) = 1 - P(B|A^c) = 0.3$. Therefore $P(A^c|B^c) = (0.3)(0.7)/0.39 = 0.54$.

2. **(a)** Define the following events:

$$T = \{\text{Exactly 2 white balls were chosen}\} \qquad W_F = \{\text{Ball from urn F was white}\}$$
$$W_E = \{\text{Ball from urn E was white}\} \qquad W_G = \{\text{Ball from urn G was white}\}$$

The probability we seek is $P(W_E \mid T) = P(W_E T)/P(T)$. Since the picks from the three baskets are all independent, we can write

$$\begin{aligned} P(T) &= P(W_E W_F W_G^c) + P(W_E W_F^c W_G) + P(W_E^c W_F W_G) \\ &\overset{a}{=} \frac{3}{6} \cdot \frac{8}{12} \cdot \frac{3}{4} + \frac{3}{6} \cdot \frac{4}{12} \cdot \frac{1}{4} + \frac{3}{6} \cdot \frac{8}{12} \cdot \frac{1}{4} = \frac{9}{24} \end{aligned}$$

$$\begin{aligned} P(W_E T) &= P(W_E W_F W_G^c) + P(W_E W_F^c W_G) \\ &\overset{b}{=} \frac{3}{6} \cdot \frac{8}{12} \cdot \frac{3}{4} + \frac{3}{6} \cdot \frac{4}{12} \cdot \frac{1}{4} = \frac{7}{24}, \end{aligned}$$

where (a) and (b) follow from independence. Therefore $P(W_E \mid T) = \dfrac{7/24}{9/24} = \dfrac{7}{9}$.

(b) Define the following events:

$$W = \{\text{Ball picked was white}\} \qquad P_F = \{\text{Ball picked was from urn F}\}$$
$$P_E = \{\text{Ball picked was from urn E}\} \qquad P_G = \{\text{Ball picked was from urn G}\}$$

The probability we seek is $P(P_E \mid W) = P(P_E W)/P(W)$, which, by Bayes' Rule, can be written as $P(P_E \mid W) = \dfrac{P(W \mid P_E)P(P_E)}{P(W)}$. From the theorem of total probability, we can write

$$P(W) = P(W \mid P_E)P(P_E) + P(W \mid P_F)P(P_F) + P(W \mid P_G)P(P_G).$$

Now $P(P_E) = P(P_F) = P(P_G) = 1/3$, since the ball was chosen with equal probability from any of the baskets; also $P(W \mid P_E) = 3/6$, $P(W \mid P_F) = 8/12$, $P(W \mid P_G) = 1/4$. This gives

$$P(W) = \frac{3}{6} \cdot \frac{1}{3} + \frac{8}{12} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{3} = \frac{17}{36}.$$

Therefore, $P(P_E \mid W) = \dfrac{3/6 \cdot 1/3}{17/36} = \dfrac{6}{17}$.

3. Ternary Channel: The numbers given in the table are all the conditional probabilities of receiving $s_j, j = 1, 2, 3$ given that you send s_i, $i = 1, 2, 3$, i.e. $P[s_j \text{ received} \mid s_i \text{ sent}]$. Therefore, the probability that the signal s_j is received can be calculated using the theorem of total probability as

$$P[s_j \text{ received}] = \sum_{i=1}^{3} P[s_j \text{ received} \mid s_i \text{ sent}] \cdot P[s_i \text{ sent}].$$

Since $P[s_i \text{ sent}] = 1/3$ for all i, the required probabilities above are found to be

	s_1	s_2	s_3
$P[s_j \text{ received}]$	0.29	0.36	0.35

Now, we compute $P[s_i \text{ sent} \mid s_j \text{ received}]$. For example,

$$P[s_1 \text{ sent} \mid s_1 \text{ received}] = \frac{P[s_1 \text{ received} \mid s_1 \text{ sent}] \cdot P[s_1 \text{ sent}]}{P[s_1 \text{ received}]} = \frac{0.8 \times 1/3}{0.29} = 0.9195.$$

In this fashion, we can form a table similar to the one given in the problem, but consisting of the inverse conditional probabilities $P[s_i \text{ sent} \mid s_j \text{ received}]$:

		Send, i		
		s_1	s_2	s_3
	s_1	0.9195	0.0575	0.0230
Receive, j	s_2	0.0926	0.8333	0.0741
	s_3	0.0952	0.0476	0.8571

4. **(a)** Define the following events:

$$\begin{aligned} A &= \{\text{an airborne narbee being involved in an accident}\} \\ S &= \{\text{narbee driver is a sober Nargian}\} \\ D &= \{\text{narbee driver is a drunk Nargian}\} \\ X &= \{\text{narbee driver is an alien}\} \end{aligned}$$

Let us denote the probability that a driver is a sober Nargian by $P(S) = p$. Then from the problem, $P(D) = 2P(S) = 2p$ and since $P(S) + P(D) + P(X) = 1$, we have $P(X) = 1 - 3P(S) = 1 - 3p$. Now using the theorem of total probability, we have

$$\begin{aligned} P(A) &= P(A \mid S)P(S) + P(A \mid D)P(D) + P(A \mid X)P(X) \\ \Rightarrow 0.59 &= 0.2p + 0.8(2p) + 0.5(1 - 3p) \Longrightarrow p = \frac{3}{10} \end{aligned}$$

Therefore, the probability that a randomly chosen driver is an alien is $1 - 3 \times \frac{3}{10} = 1/10$. And also, $P(D) = 6/10$.

(b) We want to find the probability $P(D \mid A^c)$. We can use Bayes' rule for this

$$P(D \mid A^c) = \frac{P(A^c \mid D)\, P(D)}{P(A^c)} = \frac{0.2 \times 0.6}{0.41} = \frac{12}{41} \approx 0.293$$

5. Let A_i denote the event that the student passes the i-th exam. Then we are given the following (conditional) probabilities:

$$P(A_1) \;=\; 0.9 \qquad P(A_2|A_1) \;=\; 0.8 \qquad P(A_3|A_1A_2) \;=\; 0.7$$

Based on this information, we can answer the questions in this problem as follows.

(a) Let $\mathcal{A}$ be the event that the student passes all three exams. Then, using the chain rule for conditional probabilities, we obtain

$$P(\mathcal{A}) = P(A_1A_2A_3) = P(A_3|A_1A_2)P(A_2|A_1)P(A_1) = (0.7)(0.8)(0.9) = 0.504$$

(b) Now let B_i denote the event that the student fails the i-th exam. Notice that $B_1 = A_1^c$, but $B_2 \neq A_2^c$. This is so because the events A_2 and B_2 can occur only if the event A_1 occurs, namely if the student passes the first exam. Thus the events A_2 and B_2 form a partition of the event A_1, not of the sample space Ω.

Here, we are asked for the conditional probability of the event B_2, given the event $\mathcal{A}^c$. Notice that if B_2 occurs, namely the student fails the second exam, then she cannot pass all three exams and $\mathcal{A}^c$ definitely occurs. Thus $P(\mathcal{A}^c|B_2) = 1$. Using Bayes inversion formula, we obtain:

$$P(B_2|\mathcal{A}^c) = \frac{P(\mathcal{A}^c|B_2)P(B_2)}{P(\mathcal{A}^c)} = \frac{P(B_2)}{1 - P(\mathcal{A})} = \frac{P(B_2)}{0.496} \tag{1}$$

Now let us compute the probability $P(B_2)$ that the student fails the second exam, by conditioning on the event A_1. Notice that if the event $A_1^c = B_1$ occurs, then the student does not take the second exam, and neither of the events A_2 or B_2 occurs. Thus $P(B_2|A_1^c) = 0$. On the other hand, since A_2 and B_2 partition the event A_1, we can conclude that $P(B_2|A_1) = 1 - P(A_2|A_1) = 0.2$. We therefore obtain

$$P(B_2) = P(B_2|A_1)P(A_1) + P(B_2|A_1^c)P(A_1^c) = (0.2)(0.9) + (0)(0.1) = 0.18$$

using the theorem of total probability. This is all we need. Substituting this result into Eq. (1), we find that $P(B_2|\mathcal{A}^c) = (0.18)/(0.496) = 45/124 \simeq 0.3629$.

6. Let us denote the blue-eyed gene by b and the brown-eyed gene by B. Since Smith's sister has blue eyes, she must have bb; hence both their parents must have the gene-pair bB, since they are both brown eyed. Let Y denote the event that Smith has brown eyes.

(a) Define the events X_i, $i = 0, 1, 2$, that Smith has exactly i brown-eyed genes (B's). Then $X_0 = \{bb\}$, $X_1 = \{bB, Bb\}$, $X_2 = \{BB\}$, and $Y = \{bB, Bb, BB\} = X_1 \cup X_2$, where we can assume that the first gene in a pair comes from the mother and the second from the father. We want the probability $P(X_1 \mid Y)$.

$$P(X_1 \mid Y) = \frac{P(X_1Y)}{P(Y)} = \frac{2/4}{3/4} = \frac{2}{3}.$$

Similarly, $P(X_2 \mid Y) = 1/3$, and $P(X \mid Y) = 0$.

(b) Smith's wife has bb (blue eyes). Let Z_1 denote the event that their first child also has bb. The child definitely gets a 'b' from its mother, and so the only randomness comes from Smith's contribution. We will use the theorem of total probability applied to the probability measure $P(\cdot \mid Y)$ to condition Z_1 on the events X_1 and X_2 (since these two events partition Y, we don't have to consider X_0)

$$P(Z_1 \mid Y) = P(Z_1 \mid X_1Y)P(X_1 \mid Y) + P(Z_1 \mid X_2Y)P(X_2 \mid Y) = \frac{1}{2} \cdot \frac{2}{3} + 0 \cdot \frac{1}{3} = \frac{1}{3}$$

(c) Let Z_2 denote the event that their second child has blue eyes. We want to calculate the probability $P(Z_2^c \mid YZ_1^c)$. Again, the only randomness is Smith's contribution. Using the definition of conditional probability and the theorem of total probability as in the previous part, we get

$$P(Z_2^c \mid YZ_1^c) = \frac{P(Z_2^cZ_1^c \mid Y)}{P(Z_1^c \mid Y)} = \frac{P(Z_2^cZ_1^c \mid X_1Y)P(X_1 \mid Y) + P(Z_2^cZ_1^c \mid X_2Y)P(X_2 \mid Y)}{1 - P(Z_1 \mid Y)}$$
$$= \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} + 1 \cdot \frac{1}{3}}{\frac{2}{3}} = \frac{3}{4},$$

where $P(Z_2^cZ_1^c \mid X_1Y) = 1/2 \times 1/2$ because given that Smith has exactly one B, the probability that any child gets this is $1/2$; $P(Z_2^cZ_1^c \mid X_2Y) = 1$ because if Smith has BB then his children are automatically brown-eyed.

7. **(a)** Let A_1 and A_2 respectively denote the events that the first (Dilbert) and second (Wally) players initially pick the correct curtain. A_1 and A_2 are disjoint events. Let $A = A_1 \cup A_2$ be the event that one of the players picked the correct curtain, and let B denote the event that the final choice of the player *not* eliminated is the correct curtain. Clearly $P(A) = 2/3$ and hence, $P(A^c) = 1/3$.

Stick to pick strategy: If the player not eliminated sticks to the original choice, then $P(B \mid A) = 1$, while $P(B \mid A^c) = 0$. Hence $P(B) = P(B \mid A) \cdot P(A) + P(B \mid A^c) \cdot P(A^c) = 2/3$.

Ditch and switch strategy: If the player switches, then $P(B \mid A) = 0$, while $P(B \mid A^c) = 1$. Hence, $P(B) = 1/3$ now.
Thus it is better to stay with the original pick in this game. Switching at random gives a probability of $1/2$ of winning.

(b) Assuming that both players stick to their original choices, the i^{th} player, $i = 1, 2$, will win if and only if event A_i occurs, i.e., $P(i^{th}$ player wins prize$) = P(A_i)$. Obviously, $P(A_1) = 1/3$, and hence, $P(A_2) = P(A) - P(A_1) = 1/3$. Both have equal probabilities of winning. The remaining $1/3$ of the time A^c occurs and neither wins.

(c) **Extra credit:** The method for solving this problem is similar to the "3-door Monty Hall" problem that we solved in class. Except we now have two decisions to take into account, and hence 4 possibilities — Stay-Stay (St-St), Stay-Switch (St-Sw), Switch-Stay (Sw-St), and Switch-Switch (Sw-Sw). Define the following events:

$$\begin{aligned} R &= \{\texttt{Dilbert's final choice is correct}\} \\ R_1 &= \{\texttt{Dilbert's first choice is right}\} \\ R_2 &= \{\texttt{Dilbert's second choice is right}\} \end{aligned}$$

We will partition the sample space into 4 mutually exhaustive events: R_1R_2, $R_1^cR_2$, $R_1R_2^c$, and $R_1^cR_2^c$. Now use the theorem of total probability to get

$$\begin{aligned} P(R) = P(R \mid R_1R_2)P(R_1R_2) + P(R \mid R_1^cR_2)P(R_1^cR_2) + P(R \mid R_1R_2^c)P(R_1R_2^c) \\ + P(R \mid R_1^cR_2^c)P(R_1^cR_2^c) \end{aligned}$$

The above probabilities are tabulated in the table below under each of the 4 scenarios possible.

	St-St	St-Sw	Sw-St	Sw-Sw
$P(R_1R_2)$	$\frac{1}{4}$	$\frac{1}{4}$	0	0
$P(R \mid R_1R_2)$	1	0	x	x
$P(R_1^cR_2)$	0	0	$\frac{3}{8}$	$\frac{3}{8}$
$P(R \mid R_1^cR_2)$	x	x	1	0
$P(R_1R_2^c)$	0	0	$\frac{1}{4}$	$\frac{1}{4}$
$P(R \mid R_1R_2^c)$	x	x	0	1
$P(R_1^cR_2^c)$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{8}$	$\frac{3}{8}$
$P(R \mid R_1^cR_2^c)$	0	1	0	1
$P(R)$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{8}$	$\frac{5}{8}$

The "x" in a row indicates that the conditioning event is the null event. For example, if Dilbert switches after his first pick (no matter what he does on his second choice) it is impossible that both his first and second choices are right, i.e., $P(R_1R_2) = 0$, and so it makes no sense to condition the event R on the event R_1R_2.
Thus we see that Dilbert's best policy is to Stay-Switch, i.e., pick a curtain, stay with his pick after Monty opens the first curtain, then switch after Monty opens a curtain for the second time.
If Monty had 5 curtains on stage and repeated the game, except now Dilbert had 3 stages at which he could make a choice, you can work out and see that his best policy is to Stay-Stay-Switch. And this generalizes to n curtains too — Dilbert's best policy is to Stay-Stay-$\cdots(n-1)$ times $\cdots$-Switch.