University of Illinois | Solutions: Problem Set 3 | ECE 313 Fall 1998

1. (a) The right part of the inequality is an extension of one of the properties of a probability measure derived in class. We know that for any two sets, A and B,

$$P(A \cup B) = P(A) + P(B) - P(AB) \leq P(A) + P(B).$$

Setting $D = B \cup C$, we get

$$P(A \cup B \cup C) = P(A \cup D) \leq P(A) + P(D) = P(A) + P(B \cup C) \leq P(A) + P(B) + P(C).$$

For the other part of the inequality: since $A \subset A \cup B \cup C$, we know that $P(A) \leq P(A \cup B \cup C)$. Similar $P(B) \leq P(A \cup B \cup C)$ and $P(C) \leq P(A \cup B \cup C)$. Adding these 3 inequalities gives $P(A) + P(B) + P(C) \leq 3P(A \cup B \cup C)$, and hence, the left part of the inequality.

(b) Bonferroni's inequality: There are many ways of proving this. One method is the following. From Part (a), we know that $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$. Similarly, we can apply this inequality to the complements, A^c, B^c, C^c and get $P(A^c \cup B^c \cup C^c) \leq P(A^c) + P(B^c) + P(C^c)$. Using De Morgan's laws, the left hand side of the above inequality is just $P\left((ABC)^c\right) = 1 - P(ABC)$. The right hand side is simply $3 - (P(A) + P(B) + P(C))$. Therefore we have

$$\begin{aligned} 1 - P(ABC) &\leq 3 - (P(A) + P(B) + P(C)) \\ \Rightarrow (P(A) + P(B) + P(C)) - 2 &\leq P(ABC), \end{aligned}$$

completing the proof.

(c) Generalized Bonferroni's inequality: This is typical of the types of mathematical induction proofs you will encounter or have to produce in future. We first establish the inequality for $n = 1$. For this case, the inequality reduces to showing that $P(A_1) \geq P(A_1)$, which is trivially true (In fact, we proved it to be true for $n = 3$ above, and $n = 2$ follows along similar lines). Now assume that the assertion (in this case, the inequality) is true for all n upto $n = k - 1$. The method of mathematical induction involves showing that the assertion is true for $n = k$, and since it was true for $n = 1$, and it is true for all n upto $n = k - 1$, it is true for all n. So, we have to show that

$$P(A_1 A_2 \ldots A_k) \geq P(A_1) + P(A_2) + \cdots P(A_k) - (k - 1)$$

Denote the set $A_1 A_2 A_3 \ldots A_{k-1}$ by B. Then the right hand side above becomes $P(BA_k)$. Now we have

$$\begin{aligned} P(BA_k) &\overset{a}{\geq} P(B) + P(A_k) - 1 \\ &= P(A_1 A_2 A_3 \ldots A_{k-1}) + P(A_k) - 1 \\ &\overset{b}{\geq} P(A_1) + P(A_2) + \cdots P(A_{k-1}) - (k - 2) + P(A_k) - 1 \\ &= P(A_1) + P(A_2) + \cdots P(A_k) - (k - 1), \end{aligned}$$

proving the claim. Inequalities (a) and (b) above follow from the induction hypothesis on $n = 2$ and $n = k - 1$ respectively.

2. $P(A \cup (B^c \cup C^c)^c) = P(A \cup BC)$.
   From this, we can calculate this probability in the following three cases:

   (a) B, C mutually exclusive $\Longrightarrow BC = \emptyset$. Therefore $P(A \cup BC) = P(A) = 3/7$.

   (b) $P(A \cup BC) = P(A) + P(BC) - P(ABC)$, from the formula for the probability of the union of two sets. Since $P(AC) = 0$, it follows that $P(ABC) = 0$ (why?) and therefore, plugging in the given probabilities for the other events, we get $P(A \cup BC) = 1/2 + 1/3 = 5/6$.

   (c) $P(A \cup (B^c \cup C^c)^c) = 1 - P(A^c \cap (B^c \cup C^c)) = 1 - P(A^cB^c \cup A^cC^c) = 4/7$, using De Morgan's laws.

3. (a) Let us group A and B together as a new "person". The number of "people" to be arranged in line is now $N - 1$, and the number of ways of doing this is $(N-1)!$. But the number of ways of permuting A and B within their group is 2, and so the total number of arrangements keeping A and B together is $2 \cdot (N-1)!$. Therefore the probability that A and B are together is $\dfrac{2 \cdot (N-1)!}{N!} = \dfrac{2}{N}$.

   (b) When the N people are arranged in a circle, the total number of arrangements goes down (because of circular symmetry) to $(N-1)!$. One way to think about this is the following: if you fix a particular order of people, no matter how you rotate them around the table, their relative positions (and hence their order) remains the same (convince yourselves of this by taking an example with a small enough N), i.e., the number of effective people now is $N - 1$. Now again, we group A and B together to form one "person" and the number of "people" to arrange now is $N - 2$. And as before, the number of ways this can be done is $2 \cdot (N-2)!$. Therefore the probability that A and B are together is $\dfrac{2 \cdot (N-2)!}{(N-1)!} = \dfrac{2}{N-1}$.

4. The key behind this problem is obviously to compare the 3 different pairs of spinners. Lets start with spinners a and b. We denote the event that spinner a lands on the number 9 by $\{a = 9\}$ and so on. The probability of spinner a winning over spinner b is

$$P\{a = 9\} + P\{a = 5, b = 4\} + P\{a = 5, b = 3\} = \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{5}{9}.$$

   Therefore the probability of spinner b winning over spinner a is $4/9$.

   Comparing spinners b and c, we see that the probability of spinner b winning over spinner c is

$$P\{b = 8\} + P\{b = 4, c = 2\} + P\{b = 3, c = 2\} = \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{5}{9},$$

   and so the probability of spinner c winning over spinner b is $4/9$. Likewise, comparing spinners c and a, we see that the probability of c winning over a is $5/9$.

   Hence, it is better to be player B because whichever spinner A chooses, B can always choose a spinner that gives him/her a better probability of winning.

5. (a) Let E denote the event that the bridge hand is void in at least one suit, and E_1, E_2, E_3, E_4 denote the event that the hand is void in Spades, Hearts, Clubs and Diamonds respectively. Then, denoting the 4 suits by their first letters,

$$E = \{\texttt{void in S OR void in H OR void in C OR void in D}\} = E_1 \cup E_2 \cup E_3 \cup E_4.$$

Of course, these are not mutually exclusive sets, so

$$P(E) = \sum_{i=1}^{4} P(E_i) - \sum_i \sum_{j<i} P(E_i E_j) + \sum_i \sum_{j<i} \sum_{k<j} P(E_i E_j E_k) - \sum_i \sum_{j<i} \sum_{k<j} \sum_{l<k} P(E_i E_j E_k E_l)$$

Obviously $P(E_i E_j E_k E_l) = \Phi$ for $i \neq j \neq k \neq l$ since a bridge hand can't be void in *all four suits*! now, $P(E_i E_j E_k)$ is the probability that the hand is void in 3 particular suits, say S, H, and D. The probability of that event is $1/\binom{52}{13}$, since there is only one way in which you can get all 13 Diamonds. The number of such events is $\binom{4}{3}$, the number of ways of picking the 3 voided suits out of 4. Similarly, $P(E_i E_j)$, the probability of being void in two particular suits is $\binom{26}{13}/\binom{52}{13}$, and the number of such pairs of suits is $\binom{4}{2}$. The probability of being void in a particular suit, $P(E_i) = \binom{39}{13}/\binom{52}{13}$, and the number of ways of picking that suit is 4. Therefore

$$P(E) = \frac{\binom{4}{1}\binom{39}{13} - \binom{4}{2}\binom{26}{13} + \binom{4}{3}}{\binom{52}{13}} = 0.051.$$

(b) Let F denote the event that the bridge hand contains at least one card from each suit. Then obviously $F = E^c$, so that $P(F) = 1 - P(E) = 0.949$.

6. **Matching Problem**: This is also called the problem of *derangements*.

(a) The probability that the router redirects all packets wrong is the same as the "matching problem" discussed in Ross, p. 44. We first calculate the probability of the complementary event — at least one station receives its packet, and call this event G. Also define the following events

$$B_i = \{\text{Station } i \text{ gets the } \textit{right} \text{ packet}\}$$

Now $P(G) = P(B_1 \cup B_2 \cup \cdots B_8)$, and from the formula for the probability of the union of events we know that

$$P(B_1 \cup B_2 \cup \cdots B_8) = \sum_i P(B_i) - \sum_{i_1 < i_2} P(B_{i_1} B_{i_2}) + \sum_{i_1 < i_2 < i_3} P(B_{i_1} B_{i_2} B_{i_3}) - \cdots + (-1)^9 P(B_1 B_2 \cdots B_8), \quad (1)$$

and from Ross [pp. 44–45], we know that this is equal to $1 - \frac{1}{2!} + \frac{1}{3!} - \cdots - \frac{1}{8!} = 0.6321$. Therefore $P(G^c) = 1 - P(G) = 0.3679$. As the number of stations (and packets) increases, we will see this probability approaching $1/e \approx 0.3679$ (it would appear that 8 is large enough!).

(b) We first define the events we are interested in:

$$E = \{\text{The router gets at least 6 packets routed correctly}\}$$
$$E_i = \{\text{Router gets exactly } i \text{ packets routed correctly}\}$$

Now, we see that $E = E_6 \cup E_8$ (what about E_7?). The probability of routing all 8 packets correctly is the probability of getting one particular arrangement out of 8! possible arrangements of packets, and therefore $P(E_8) = \frac{1}{8!}$. Similarly, $P(E_6) = \left(\binom{8}{6} \cdot 1\right)/8!$, where $\binom{8}{6}$ is the number of ways of choosing 6 stations out of 8, and 1 is the number of ways in which the remaining two stations can get their packets wrong (they have to get their packets exchanged). Therefore,

$$P(E) = P(E_6) + P(E_8) = \frac{\binom{8}{6} + 1}{8!} = \frac{29}{40320}$$

(c) We "renumber" the stations $A, B, \ldots, H$ as $1, 2, \ldots, 8$, and define the following events:

$$A = \{\text{Station 1 gets the wrong packet}\}$$
$$B = \{\text{No station gets it's desired packet}\}$$

It is clear that the probability we seek is $P(B \mid A) = P(BA)/P(A) = P(B)/P(A)$, since $B \subset A$. It is further easy to see that the probability that station A gets the wrong packet, $P(A) = 7/8$. Therefore, the desired probability

$$P(B \mid A) = P(B)/P(A) \approx 0.36788 \times 8/7 \approx 0.4204$$

7. Define the following events:

$$A_k = \left\{\text{The first ace is the } k^{th} \text{ card to appear}\right\}$$
$$B = \{\text{The next card is the ace of spades}\}$$
$$C = \{\text{The next card is the two of clubs}\}$$

(i) $P(B \mid A_3) = P(A_3B)/P(A_3)$, and $P(A_3) = 48/52 \cdot 47/51 \cdot 4/50$. Now, $P(A_3B)$, the probability that the first ace to appear is the third card *and* that the next card is the ace of spades, is $P(A_3B) = \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{3}{50} \cdot \frac{1}{49}$. Therefore

$$P(B \mid A_3) = \frac{\frac{48}{52} \cdot \frac{47}{51} \cdot \frac{3}{50} \cdot \frac{1}{49}}{\frac{48}{52} \cdot \frac{47}{51} \cdot \frac{4}{50}} = \frac{3}{4 \cdot 49} = \frac{3}{196}$$

(ii) Similarly, $P(C \mid A_3) = P(A_3C)/P(A_3)$, and $P(A_3C) = \frac{47}{52} \cdot \frac{46}{51} \cdot \frac{4}{50} \cdot \frac{1}{49}$. Therefore,

$$P(C \mid A_3) = \frac{\frac{47}{52} \cdot \frac{46}{51} \cdot \frac{4}{50} \cdot \frac{1}{49}}{\frac{48}{52} \cdot \frac{47}{51} \cdot \frac{4}{50}} = \frac{46}{49 \cdot 48} = \frac{23}{1176}$$

If the events are changed to reflect the fact that the first ace to appear is the k^{th} card, then we have:

$$P(A_k) = \frac{48P_{k-1} \cdot 4}{52P_k},$$

because the first $k-1$ cards have to be chosen out of 48 (no aces) and the k^{th} card can be chosen in 4 ways. Similarly,

$$P(A_kB) = \frac{48P_{k-1} \cdot 3 \cdot 1}{52P_{k+1}}$$

because now we are choosing $k+1$ cards, and the k^{th} card cannot be the Ace of Spades, while the $(k+1)^{st}$ card has to be the Ace of Spades. Notice in both cases we don't care about the remaining deck of cards. So now $P(B \mid A_k) = P(A_kB)/P(A_k)$ so

$$P(B \mid A_k) = \frac{\dfrac{48P_{k-1} \cdot 3 \cdot 1}{52P_{k+1}}}{\dfrac{48P_{k-1} \cdot 4}{52P_k}} = \frac{3}{4(52-k)} = \frac{3}{128} \text{ if } k = 20$$

Likewise $P(A_kC) = 47P_{k-1} \cdot 4 \cdot 1/52P_{k+1}$, and so for $k = 20$, after some simplification you get

$$P(C \mid A_k) = \frac{\dfrac{47}{52} \cdot \dfrac{46}{51} \cdot \dfrac{45}{50} \cdots \dfrac{29}{34} \cdot \dfrac{4}{33} \cdot \dfrac{1}{32}}{\dfrac{48}{52} \cdot \dfrac{47}{51} \cdot \dfrac{46}{50} \cdots \dfrac{30}{34} \cdot \dfrac{4}{33}} = \frac{29}{32 \cdot 48} = \frac{29}{1536}$$

8. **[Extra Credit]:** This problem can be worked in two different (maybe more) ways. The hard solution first: this uses conditioning. But it is more subtle than the previous question because we have to condition on each of the events, $A_k = \{\text{The first ace to appear is the } k^{th} \text{ card}\}$. Let us further define the events

$$B = \{\text{The } \textit{next} \text{ card is the ace of spades}\}$$
$$C = \{\text{The } \textit{next} \text{ card is the two of clubs}\}$$

There are two ways to approach the problem. The first way is through the theorem of total probability. This states that $P(B) = \sum_k P(B \mid A_k)P(A_k) = \sum_k P(BA_k)$. In this case, it is easier to calculate $P(BA_k)$ than $P(B \mid A_k)$. The event BA_k is the event that the first ace is the k^{th} card and the $(k+1)^{st}$ card is the ace of spades. Similar to part (a),

$$P(BA_1) = \frac{3}{52} \cdot \frac{1}{51}$$
$$P(BA_k) = \frac{48}{52}\frac{47}{51} \cdots \frac{50-k}{54-k}\frac{3}{53-k}\frac{1}{52-k}, \quad 2 \le k \le 49$$
$$P(BA_k) = 0, \qquad k = 50, 51, 52$$

Convince yourselves of the limits for k. Therefore,

$$P(B) = \frac{3}{52}\frac{1}{51} + \sum_{k=2}^{49} \frac{48}{52}\frac{47}{51}\frac{46}{50} \cdots \frac{50-k}{54-k}\frac{3}{53-k}\frac{1}{52-k}$$
$$= \sum_{k=1}^{49} \frac{3 \cdot 48!}{52!}(51-k)(50-k) = \frac{3 \cdot 48!}{52!} \sum_{k=1}^{49} k(k+1),$$

where the last equality follows by substituting $(50-k)$ with k and recalculating the limits. We now use the following formulae for sums: $\sum_{i=1}^{N} k = N(N+1)/2$; $\sum_{i=1}^{N} k^2 = N(N+1)(2N+1)/6$. The above sum now evaluates as $\sum_{k=1}^{N} k(k+1) = N(N+1)(N+2)/3$, and from this, for our case where $N = 49$, we get

$$P(B) = \frac{3 \cdot 48!}{52!} \frac{49 \cdot 50 \cdot 51}{3} = \frac{1}{52}$$

It's amazing how this simplifies!

$P(C)$ can be calculated in a similar manner, $P(C) = \sum_k P(CA_k)$.

$$\begin{aligned} P(CA_1) &= \frac{4}{52} \cdot \frac{1}{51} \\ P(CA_k) &= \frac{47}{52} \frac{46}{51} \cdots \frac{49-k}{54-k} \frac{4}{53-k} \frac{1}{52-k}, \quad 2 \le k \le 48 \\ P(CA_k) &= 0, \qquad k = 49, 50, 51, 52 \end{aligned}$$

As before, convince yourselves of the limits for k. Therefore,

$$\begin{aligned} P(C) &= \frac{4}{52} \frac{1}{51} + \sum_{k=2}^{48} \frac{47}{52} \frac{46}{51} \frac{45}{50} \cdots \frac{49-k}{54-k} \frac{4}{53-k} \frac{1}{52-k} \\ &= \sum_{k=1}^{48} \frac{4 \cdot 47!}{52!} (51-k)(50-k)(49-k). \end{aligned}$$

The above summation might seem daunting, but it's actually easy. The sum can be rewritten as $\sum_{k=1}^{48}(51-k)(50-k)(49-k) = \sum_{k=1}^{48} k(k+1)(k+2)$. (This is obtained, as above, by substituting $49-k$ with k and recalculating the limits.) This sum is obtained by using the formulae for summations listed above along with one more: $\sum_{i=1}^{N} k^3 = N^2(N+1)^2/4$. The result is quite interesting:

$$\sum_{k=1}^{N} k(k+1)(k+2) = \frac{N(N+1)(N+2)(N+3)}{4},$$

and therefore, in our case, where $N = 48$, the summation gives $\frac{48 \cdot 49 \cdot 50 \cdot 51}{4}$. Plugging this back in the formula for $P(C)$, we see again that $P(C) = 1/52$. (!!) So, it doesn't matter which card comes after the first ace — the probability of it appearing is the same!

Simple solution: After working through those laborious summations, the thought occurs that if the final answer is so simple, there has to be a simpler way of obtaining it. There is: consider the first Ace and the Ace of Spades as one unit (just as in Problem 3). The number of permutations of the 51 "units" now is 51! and the total number of permutations of 52 cards is 52!. Therefore the probability that the Ace of Spades immediately follows the first ace is $51!/52! = 1/52$. Similarly, the probability that the 2 of Clubs immediately follows the first Ace is also $1/52$ (same reasons). Simple, eh?