ECE 313: Problem Set #2

Assigned: September 2, 1998
Due: September 9, 1998
Reading: Ross, Chapters 1 and 2

1. Consider an experiment with a finite sample space Ω consisting of n outcomes, all equally likely.

(**a**) Show that there are 2^n distinct events defined on this sample space. Thereby, prove the identity

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

(**b**) Show that 2^{n-1} events contain an odd number of outcomes and 2^{n-1} events contain an even number of outcomes.

(**c**) Find the "average probability" of an event by adding up the probabilities of all the 2^n events and dividing the result by the total number 2^n of events.

(**d**) How many of the 2^n events have probability exactly equal to the "average probability" that you found in part (**c**)?

2. [**15pts**] An ice cream store has five basic flavors of icecream — butterscotch, chocolate, mocha, strawberry, and vanilla; and three basic toppings — almonds, fudge, and peanut-butter. It creates specialty icecreams by: using one basic flavor and one or more toppings OR two or more basic flavors and zero or more toppings (if you want toppings, they have to be *on top* of the scoops of icecream).

(**a**) How many specialty icecreams combos can be obtained by using three distinct basic flavors and any number of toppings? How many of these contain chocolate? How many of these contain either chocolate or strawberry but not both (yuck!)?

(**b**) Calvin, Suzie, and Hobbes are in line at the icecream store. Suzie is picky and wants three distinct icecream flavors and two different toppings, but will not eat vanilla and strawberry together UNLESS one of the toppings is fudge. How many choices does she have?

(**c**) How many distinct **specialty** icecreams can the store offer?

(**d**) Calvin, who is behind Suzie, wants three scoops (he doesn't care that they are all different or not — of course, {two scoops of chocolate and one of vanilla} is different from {one scoop of chocolate and two of vanilla}, right?) and two toppings (again, he doesn't care that they are different or not). How many different icecreams can he get? (Be careful on this one.)

(**e**) [**Extra credit, 10 pts**] Much to Calvin's chagrin, Hobbes and Suzie decide to share the Scoop-Splurge, which is 6 scoops of icecream (any flavors) and three toppings. They don't care which six scoops they get (repeats are fine), **as long as** chocolate and mocha are two of them AND strawberry **is not** one of them; and they don't care which three toppings they get (again, repeats are okay) **as long as** one of them is fudge. How many choices do they have?

Optional for zero additional credit: Identify the store!

3. Express each of the following events in terms of the events A, B, and C, and the operations of complementation, union, and intersection:

(a) at least one of the events A, B, C occurs;
(b) at most one of the events A, B, C occurs;
(c) none of the events A, B, C occurs;
(d) all three events A, B, C occur;
(e) exactly one of the events A, B, C occurs;
(f) events A and B occur, but not C;
(g) either event A occurs, or if not then B also does not occur.

In each case draw the corresponding Venn diagrams or Karnaugh maps, whichever is more convenient.

4. Suppose that A and B are two events defined over the same sample space, with probabilities $P(A) = 3/4$ and $P(B) = 1/3$.

(a) Show that $P(A \cup B) \geq 3/4$.
(b) Show that $1/12 \leq P(AB) \leq 1/3$.
(c) Give inequalities analogous to **(a)** and **(b)** for $P(A) = 2/3$ and $P(B) = 1/2$.

5. The experiment consists of picking a student from the set of all students registered on the UIUC campus this semester. It is *not* necessary to assume that all students are equally likely to be picked, but you may make this assumption if you like.

(a) Consider the two events:

A — the student has had four years of high school science (FYS),
B — the student has had calculus in high school.

If the probability that (s)he has had at least one of FYS and calculus is 0.6, and the probability that (s)he has missed at least one of the two is 0.8, what is the probability that (s)he has had *exactly* one of the two?

(b) Let C denote the event that the student is registered in ECE313 this semester, and let events A and B and their probabilities be as in part **(a)**. If students who had had at most one of FYS and calculus did not register in EE313 this semester,

i. What is $P(A^c \cap B^c \cap C^c)$?
ii. What is the probability that the student picked is not registered in ECE313 *and* has had exactly one of FYS or calculus in high school?

(c) Using the data given in parts **(a)** and **(b)**, which of the following probabilities:

$$P(ABC),\ P(C),\ \ P(A^cBC^c),\ \ P(A),\ \ P(A^cB^cC),\ \ P(A \cup B \cup C^c)$$

can you compute? It is not necessary to actually compute each probability.

6. Let $\Omega = \{1, 2, 3, \ldots\}$ be the countably infinite sample space whose elements (outcomes) are the positive integers. For each positive integer n, define the event

$$A_n = \{\ k\ :\ k \text{ is a multiple of } n\ \}$$

(a) Find n and m such that $A_n = A_3 \cap A_4$ and $A_m = A_6 \cap A_9$.

(b) If $P(\{k\}) = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{k-1}$ find the probability of the event A_3.

Note: an exact answer of the form a/b, where a and b are integers, is required; answers of the form 0.210526... will receive no credit.