University
of Illinois

Problem Set 13

ECE 313
Fall 1998

Assigned : Friday, November 20, 1998
Due : Wednesday, December 2, 1998
Reading : Ross, Chapters 6,7

1. Three points $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are independently selected at random from the interval $[0, 2]$.

   (a) What is the probability that $\mathbf{X}_2$ lies between $\mathbf{X}_1$ and $\mathbf{X}_3$?

   (b) What is the probability that the largest of the three (the largest is not necessarily $\mathbf{X}_3$) is greater than 1?

   (c) What is the probability that the largest of the three is greater than the sum of the other two?

2. Suppose $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ are independent random variables that are each equally likely to be either 1 or 2. Find the pmf of

   (a) $\mathbf{XYZ}$

   (b) $\mathbf{XY} + \mathbf{YZ} + \mathbf{XZ}$

   (c) $\mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2$

3. If $\mathbf{U}$ is uniform on $(0, 2\pi)$ and $\mathbf{Z}$, independent of $\mathbf{U}$, is exponential with parameter $\lambda = 1$,

   (a) Find the joint density of the random variables $\mathbf{X}, \mathbf{Y}$ defined by

$$\mathbf{X} = \sqrt{2\mathbf{Z}} \cos \mathbf{U}$$
$$\mathbf{Y} = \sqrt{2\mathbf{Z}} \sin \mathbf{U}$$

   (b) Show that $\mathbf{X}$ and $\mathbf{Y}$ are independent unit normal RVs, i.e., that they are independent and each of them is distributed as $N(0, 1)$.

   (c) What is the pdf of the random variable $\mathbf{R} = \sqrt{\mathbf{X}^2 + \mathbf{Y}^2}$?

4. $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent random variables. $\mathbf{X}_1$ is uniformly distributed over $(-1, 1)$ and $\mathbf{X}_2$ is exponentially distributed with parameter $\lambda = 1$.

   (a) Find the pdfs of the RVs $\mathbf{W} = \mathbf{X}_1 + \mathbf{X}_2$ and $\mathbf{Z} = \mathbf{X}_1/\mathbf{X}_2$.

   (b) Find the joint pdf of $\mathbf{W}$ and $\mathbf{Z}$.

5. Let $(\mathbf{X}, \mathbf{Y})$ be uniformly distributed on the interior of the square with vertices at $(0,0), (0,1), (1,1)$ and $(1,0)$.

   (a) Are $\mathbf{X}$ and $\mathbf{Y}$ independent? Are they uncorrelated?

   (b) Are random variables $(\mathbf{X}+\mathbf{Y})$ and $(\mathbf{X}-\mathbf{Y})$ uncorrelated or independent?

   Now let $(\mathbf{X}, \mathbf{Y})$ be uniformly distributed on the interior of another square with vertices at $(2,0), (1,1)$, $(0,0)$ and $(1,-1)$.

   (b) Determine whether $\mathbf{X}$ and $\mathbf{Y}$ are uncorrelated or independent in this case. What does this say about random variables being independent and uncorrelated?

   (c) Are random variables $(\mathbf{X}+\mathbf{Y})$ and $(\mathbf{X}-\mathbf{Y})$ uncorrelated or independent?

6. Ross, p.298, problem 43.

7. If $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ are independent random variables having identical density functions $f(u) = e^{-u}$, $0 < u < \infty$, derive the joint distribution of $\mathbf{U} = \mathbf{X}+\mathbf{Y}$, $\mathbf{V} = \mathbf{X}+\mathbf{Z}$ and $\mathbf{W} = \mathbf{Y}+\mathbf{Z}$.

8. Let $\mathbf{X}$ and $\mathbf{Y}$ be Gaussian random variables with $E[\mathbf{X}] = 1, E[\mathbf{Y}] = -1, \text{Var}[\mathbf{X}] = 2, \text{Var}[\mathbf{Y}] = 1$ and $\rho_{X,Y} = 0.5$.

   (a) If $\mathbf{Z} = 2(\mathbf{X}+\mathbf{Y})(\mathbf{X}-\mathbf{Y})$, what is $E[\mathbf{Z}]$?

   (b) If $\mathbf{U} = 2\mathbf{X} - 3\mathbf{Y}$ and $\mathbf{W} = 2\mathbf{X} + 3\mathbf{Y}$, what is $\text{cov}(\mathbf{U}, \mathbf{W})$?

   (c) What is the joint pdf of $\mathbf{U}$ and $\mathbf{W}$?

9. **Extra Credit [10pts]:** If the random variables $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are independent, and chosen uniformly in the interval $[0,1]$, what is the probability that all of the roots of the equation $\mathbf{A}x^2 + \mathbf{B}x + \mathbf{C} = 0$ are real?