University of Illinois Fall 1998

ECE 313: Problem Set #11

Assigned: Friday, November 6
Due: Friday, November 13
Reading: Ross, sections 1,2,3 of Chapter 6

Recommended problems : These exercises are from Ross: pp. 291–294, 1–5, 10, 11–13, 21; pp. 296–297, 1–10, 13–19.

1. Suppose that two cards are drawn at random from a deck of 52 cards. Let X be the number of aces obtained and let Y be the number of queens obtained.

(**a**) Find the joint probability mass function of X and Y.

(**b**) Find the marginal probability mass function of X and that of Y.

(For those who never play card games: there are 4 aces and 4 queens in a deck of 52 cards.)

2. The discrete random variables X and Y have joint p.m.f. $p_{X,Y}(u,v)$ given by

$v \downarrow / u \rightarrow$	0	1	3	5
-1	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	0
3	$\frac{1}{6}$	$\frac{1}{12}$	0	$\frac{1}{12}$
4	0	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$

(**a**) Find the joint CDF of X and Y. Specify the value of $F_{X,Y}(u,v)$ for all u and v.

(**b**) Find the marginal probability mass functions $p_X(u)$ and $p_Y(v)$ of X and Y.

(**c**) Find $P\{X \le Y\}$ and $P\{X + Y \le 8\}$

3. Is the following function

$$F(u,v) = \begin{cases} 0 & u+v < 1 \\ 1 & u+v \ge 1 \end{cases}$$

a valid joint CDF. Why or why not? Prove your answer and show your work.

4. The random variables X and Y have joint probability density function

$$f_{X,Y}(u,v) = \begin{cases} 2e^{-(u+v)} & 0 < u < v < \infty \\ 0 & \text{otherwise} \end{cases}$$

(**a**) Sketch the uv plane with an indication of the region where $f_{X,Y}(u,v)$ is nonzero.

(**b**) Find the joint CDF of X and Y. Specify the value of $F_{X,Y}(u,v)$ for all u and v.

(**c**) Find the marginal CDFs of X and Y by setting, respectively, v and u to $+\infty$ in the answer to part (**b**).

(d) Find the marginal probability density functions of X and Y by:

i. integrating the joint probability density function

ii. differentiating the marginal CDFs found in part (**c**)

Compare the two answers obtained in (i) and (ii).

(e) Find $P\{Y > 3X\}$.

(f) For $\alpha > 0$, find $P\{X + Y \leq \alpha\}$.

(g) Use the result in part (**f**) to determine the p.d.f. of the random variable $Z = X + Y$.

5. The random variables X and Y are uniformly distributed on the unit circle $u^2 + v^2 \leq 1$.

(a) Write down the joint probability density function of X and Y.

(b) Find the marginal probability densities of X and Y.

6. [Extra Credit, 10 pts:] Let X and Y be discrete random variables taking on the real values $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_m\}$, respectively, and let $p_{i,j} = P(\{X = u_i\} \cap \{Y = v_j\})$. Show that X and Y are independent if and only if the rank of the joint p.m.f. matrix $P = [p_{i,j}]$ is equal to 1.

Linear algebra reminder: rank of a matrix = number of linearly independent rows = number of linearly independent columns.

7. [Extra Credit, 10 pts:] The joint probability density function of the variables X, Y, Z is given by

$$f_{X,Y,Z}(u,v,w) = \begin{cases} \dfrac{c \cdot (u+v+w)^2}{\sqrt{u^3 v^3 w^3}} & 0 < u, v, w < 1 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant. Find $P\{X < Y < Z\}$.

Hint: think before you start integrating.