University of Illinois

Problem Set 10

ECE 313
Fall 1998

Assigned : Wednesday, October 28, 1998
Due : Wednesday, November 4, 1998
Reading : Ross, Chapters 4,5
Additional problems : 5.3, 5.7, 5.8, 5.10, 5.12, 5.16–5.18, 5.25–5.26, 5.29, 5.32, 5.35–5.36, 5.39

1. **[15pts]** A number is chosen uniformly in $(0, 1)$.
   (a) Find the pdf/pmf of the negative of its logarithm (to base e).
   (b) Find the pdf/pmf of the second decimal of its square root.

2. Let $\mathbf{X}$ have the following probability density function : $f_X(u) = \frac{1}{2u^2}$, $|u| \geq 1$, and 0 otherwise. Find the pdf/pmf of the following random variables:
   (i) $\text{sign}(\mathbf{X}+2)$, where $\text{sign}(x) = 1$, if $x \geq 0$; $\text{sign}(x) = -1$, if $x < 0$.
   (ii) the random variable $\mathbf{Y}$, where $\mathbf{Y} = 2|\mathbf{X}|$, if $|\mathbf{X}| \leq 2$; and $\mathbf{Y} = \mathbf{X}^2$, if $|\mathbf{X}| \geq 2$.

3. Let $\mathbf{X}$ be a continuous RV with CDF $F_X(u)$.
   (a) Define another random variable $\mathbf{Y}$ by $\mathbf{Y} = F_X(\mathbf{X})$. Show that $\mathbf{Y}$ is uniformly distributed over the interval $[0, 1]$, **regardless** of what $F_X(u)$ is.
   (b) Now let's do the reverse: We wish to generate a continuous RV $\mathbf{X}$ with a specified distribution (CDF) $\mathcal{F}(u)$, and all we are given is a uniform RV on $[0, 1]$. Define the random variable $\mathbf{X}$ by $\mathbf{X} = \mathcal{F}^{-1}(\mathbf{Y})$ (remember that $\mathcal{F}$ is monotonic increasing on $[0, 1]$), where $\mathbf{Y}$ is uniform on $[0, 1]$. Show that $\mathbf{X}$ has CDF $\mathcal{F}(u)$. (This is a widely used method to generate random variables of desired distributions, for example in computer simulations.)

4. **[20pts]** A generic communication system involves the transmission of a *known* signal s over a channel that corrupts it with additive noise, which is typically modeled as a Gaussian RV
   $\mathbf{X} \sim N(0, \sigma^2)$. Thus, the received signal is a random variable given by $\mathbf{R} = s + \mathbf{X}$.
   (a) What kind of random variable is $\mathbf{R}$? Find its CDF and pdf (write down its CDF in terms of $\Phi(u)$, the CDF of a standard Gaussian random variable).
   (b) Find the pdf of the RV $\mathbf{Y} = e^{\mathbf{R}}$, and plot it ($\mathbf{Y}$ is called a *lognormal* RV since $\ln \mathbf{Y}$ is a normal (or Gaussian) RV. This random variable occurs often in wireless communications).
   (c) We now discretize the signal at the receiver by passing it through an analog-to-digital (A/D) converter that *quantizes* the waveform. The input/output relations of two different quantizers are given in the diagram below. The receiver accurately estimates the mean of $\mathbf{R}$ and quantizes it about this value. Therefore, the transition points in both cases occur symmetrically about $E[\mathbf{R}]$ (the mean of the received signal) i.e., at $E[\mathbf{R}] - c$ and $E[\mathbf{R}] + c$. You can also assume that the noise variance $\sigma^2 = 1$.

i. What kind of RV is $\mathbf{Y}$, the output of the quantizer? Calculate and sketch its CDF for both Quantizer A and Quantizer B.

ii. Now let us choose a value for c, thereby truly *designing* our quantizers. One way is to minimize the *mean squared error* between the input to the quantizer $\mathbf{R}$ and output from the quantizer $\mathbf{Y}$ with respect to c. That is, c is chosen so as to minimize $E[(\mathbf{R} - \mathbf{Y})^2]$, where the expectation is with respect to the pdf of $\mathbf{R}$. Find this value of c for Quantizers A and B.

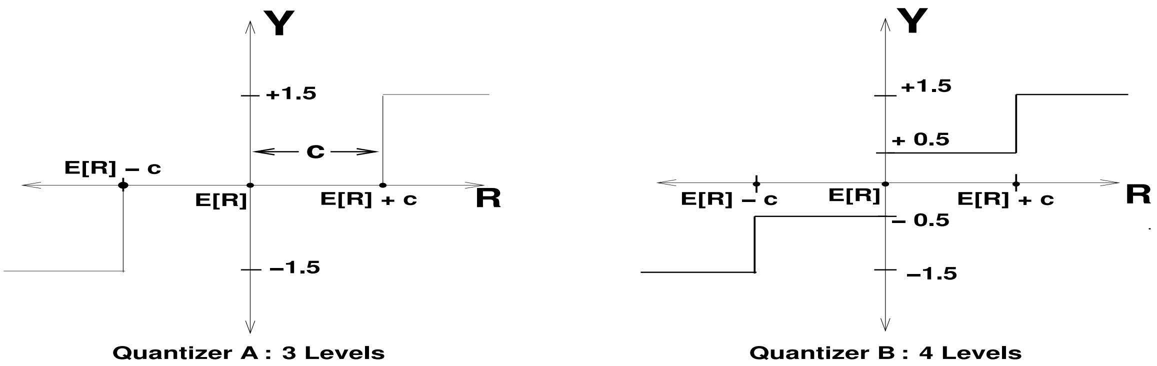

Figure 1: Three and four level Quantizers: Quantizer A is a 3-level quantizer; Quantizer B is a 4-level quantizer. The transition points in both cases are symmetrically spaced about $E[\mathbf{R}]$.

5. (a) It's Halloween time and the m&m jar in Calvin's house contains m&m's of N different colors in **equal numbers**. To keep him occupied, Calvin's mom tells him that he can eat a certain number of m&m's from the jar according to the following strategy: he should pick m&m's from the jar one by one, record their color and put them back in the jar. The moment he gets an m&m whose color he has picked before, he stops and eats one m&m for each color that he picked, **including the one repetition**. Let $\mathbf{X}$ be the RV denoting the number of m&m's that Calvin picks. What is the **maximum** value that $\mathbf{X}$ can take? Find the pmf of $\mathbf{X}$.

   (b) Calvin soon gets bored of this game and so his dad gives him this slightly more "interesting" game to play. Calvin is again supposed to start picking m&m's from the jar, recording their color and putting them back in the jar. This time, he stops only when he has seen **at least one m&m of each color**. We let $\mathbf{Y}$ be the RV denoting the number of m&m's that Calvin picks this time. What is the **minimum** value that $\mathbf{Y}$ can take? Find the pmf of $\mathbf{Y}$.

   (c) **Extra credit worth 25pts each:** In both the above games find the expected number of beans that Calvin will get to eat. (**Hint:** It might be easier in this case to *not* try to calculate this from first principles using the pmf of $\mathbf{X}$ or $\mathbf{Y}$.)