University of Illinois

Solutions: Problem Set 1

ECE 313 Spring 1998

1. (a) A normal calculator should give you answers close to 1 for decreasing values of x, so long as the precision used by the calculator is high enough. Indeed, the analytic solution to this problem is very simple and does not even need a knowledge of calculus

$$\frac{1}{\sin^2 x} - \frac{1}{\tan^2 x} = \frac{1}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x} = \mathbf{1}!$$

for any value of x and in particular, for $x \to 0$.

(b) This part is more tricky though. My cheapo calculator gives me $0.3341, 0.33333, 0.3332, 0.32$ for $x = 0.1, 0.01, 0.001, 0.0001$ and 0 for smaller values of x. Using a mathematical package like MATLAB gives 0.3333 for $x = 0.1, 0.01, 0.001, 0.0001, \ldots$. Thus, it would appear that the answer is 1/3, and indeed this is true. We give two formal proofs:

(i) Using the "familiar" Taylor series expansion, note that $\sin x = x - x^3/3! + x^5/5! - \ldots \approx x - x^3/6$ for small x. Hence, for small x,

$$\begin{aligned}\frac{1}{\sin^2 x} - \frac{1}{x^2} \approx \frac{1}{x^2}\left[\frac{1}{(1-x^2/6)^2} - 1\right] &= \frac{1}{x^2}\left[1 + 2x^2/6 + 3\left(x^2/6\right)^2 + \ldots - 1\right] \\ &= \frac{1}{3} + \frac{3x^2}{36} + \ldots,\end{aligned}$$

where we have made use of the series $\dfrac{1}{(1-y)^2} = 1 + 2y + 3y^2 + 4y^3 + \ldots$, which is valid for $|y| < 1$. This series, along with the ubiquitous series $\dfrac{1}{(1-y)} = 1 + y + y^2 + y^3 + \ldots$ are useful to know. Thus,

$$\frac{1}{\sin^2 x} - \frac{1}{x^2} \approx \frac{1}{3} + \frac{3x^2}{36} + \ldots = 1/3 \mathrm{as} \lim x \to 0.$$

(ii) Perhaps a genuinely more familiar proof uses repeated applications of L'Hôspital's rule. Note that $\dfrac{1}{\sin^2 x} - \dfrac{1}{x^2} = \dfrac{x^2 - \sin^2 x}{x^2 \sin^2 x}$ is of the form 0/0 at $x = 0$. Four applications of L'Hôspital's rule yields

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right) = \frac{8\cos(2x)}{4\cos(2x) + 12\cos(2x) + 8\cos(2x) + x(\text{some other stuff})} = \frac{8}{24} = \frac{1}{3}.$$

2. Derivatives and antiderivatives:

(i) True as per chain rule.

(ii) Also true as per chain rule.

(iii) False. Chain rule gives $\frac{d}{dx}\exp\left(f(x^2)\right)$ as $2x \cdot g(x^2)\exp\left(f(x^2)\right)$.

(iv) False. $\int g(-x)\,dx = -f(-x) + C$ (minus sign missing.)

(v) False. The antiderivative of $g(x^2/2)$ need not be related to $f(x^2/2)$ at all, as can be seen by differentiating both sides of (v). For example, take $f(x) = \ln(1+x)$ to see this.

(vi) True **only for** positive functions $f(x)$.

3. Definite integrals:

(a) Drawing a picture, you see that the integral from -1 to 1 is just the sum of the areas of two triangles, each having height 1 and base length 1. Therefore, it is easy to see that the total area $= 2 \times 1/2 = 1$. Try and draw a little sketch before you tackle most problems. Those who used formal integration principles should note that $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. Therefore

$$\int_{-1}^{1} |x|\, dx = \int_{-1}^{0} -x\, dx + \int_{0}^{1} x\, dx = \left.\frac{-x^2}{2}\right|_{-1}^{0} + \left.\frac{x^2}{2}\right|_{0}^{1} = 1.$$

(b) Substitute $y = (1-x^2)$. Then

$$\int_0^1 x(1-x^2)^{11}\, dx = -\frac{1}{2}\int_1^0 y^{11}\, dy = \frac{1}{2}\int_0^1 y^{11}\, dy = \frac{1}{24}.$$

(c) Notice that $x\exp(-x^2/2) = \frac{d}{dx}(-\exp(-x^2/2)$. Therefore

$$\int_{-\infty}^{\infty} x\exp(-x^2/2)\, dx = \int_{-\infty}^{\infty} \frac{d}{dx}(-\exp(-x^2/2)\, dx = -\exp(-x^2/2)\big|_{-\infty}^{\infty} = 0.$$

You could have also obtained the answer 0 by observing that the integrand is an odd function and therefore has to integrate to 0 over any interval that is symmetric about the origin.

4. Since an integral is the area under a curve, a non-negative function *cannot* have a negative integral! Therefore there *does not* exist a function $f(x)$ satisfying the two properties stated in the problem.

5. Two-dimensional integrals:

(a) $f(x,y) = \min(x,y)$, over the region $D = \{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$.
Drawing a three-dimensional picture is not the easiest thing, but it is worth the effort to get an idea of what the function looks like. The region of integration in the $x-y$ plane is shown in Figure 1 below.
Applying basic integration techniques and using the fact that $\min(x,y) = x$ if $x \leq y$ and $\min(x,y) = y$ if $x > y$, we get

$$\begin{aligned}\int_0^1\int_0^1 \min(x,y)\, dx\, dy &= \int_0^1\left(\int_0^y x\, dx\right) dy + \int_0^1\left(\int_y^1 y\, dx\right) dy \\ &= \int_0^1 \frac{y^2}{2}\, dy + \int_0^1 y(1-y)\, dy = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.\end{aligned}$$

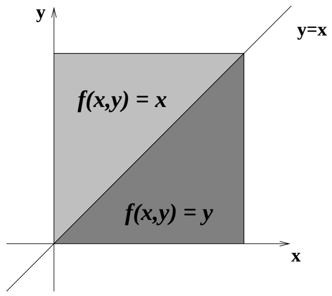

Figure 1: Problem 5 a

(b) $f(x, y) = (x^2 + y^2)^{-2}$, over the region $D = \{(x, y) : x^2 + y^2 > 1\}$.
The region of integration is shown in Figure 2 below. Changing to polar coordinates ($x = r\cos\theta, y = r\sin\theta$), the above integral can be reduced to

$$\int\int_D (x^2 + y^2)^{-2} = \int_1^\infty \int_0^{2\pi} r^{-4} \cdot r\, dr d\theta = 2\pi \int_1^\infty r^{-3}\, dr = \pi,$$

where the two-dimensional differential $dx\, dy$ transforms to $r\, dr\, d\theta$.

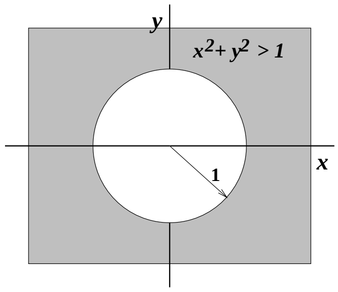

Figure 2: Problem 5 b

(c) Figure 3 below will convince you of the region of integration, $D = \{(x, y) : 0 < x + y < 1\}$, for this function:

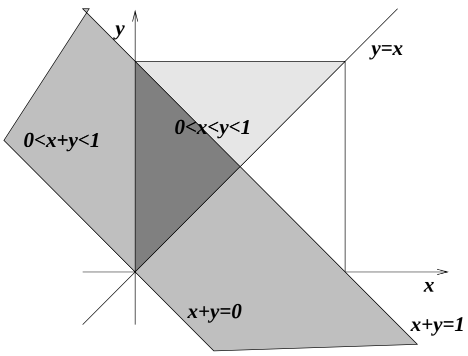

Figure 3: Problem 5 c

$$\iint_D (x+y)dy\,dx = \int_{x=0}^{1/2}\int_{y=x}^{1-x}(x+y)dy\,dx$$
$$= \int_{x=0}^{1/2} x(1-2x) + \frac{(1-x)^2-x^2}{2}\,dx = \frac{1}{6}$$

6. **Super-Calvinball:** Using simple physics ideas of relative velocity, the distance traveled by the ball, d, can be written down as a series

$$d = 100\left(\frac{2}{3}+\frac{2}{3}\cdot\frac{1}{3}+\frac{2}{3}\cdot\left(\frac{1}{3}\right)^2+\cdots\right) = 200\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^k = 200\frac{1/3}{2/3} = 100.$$

This series is called a geometric series and will figure rather prominently in future topics (and homeworks).

7. **Extra Credit:** The concepts used in this problem will be formalized in future classes, but they are intuitive enough.

(a) With four rolls of a die, the size of the sample space (which is the space of all 4-tuples of numbers from 1 to 6) is 6^4. The event we are looking for is { at least one six in 4 rolls }. The easiest way to solve this part is to consider the **complementary** event: the number of possibilities where **no** six is rolled. This number (from BPOC) is 5^4 (5 for each roll). Therefore the number of possibilities with at least one six is $6^4 - 5^4 = 671$, and required probability is $671/6^4 = 0.5177$.

(b) The same logic applies here, except we have two dice and 24 rolls. The size of the sample space (what is Ω comprised of?) $= 36^{24}$. The size of the complementary event is the number of ways in which **no** pair of sixes occurs in 24 rolls, which is 35^{24}. Therefore the number of ways in which at least one pair of sixes is rolled $= 36^{24} - 35^{24}$, and the required probability is $\frac{36^{24}-35^{24}}{36^{24}} = 0.4914$. (You would lose in the long run if you played this game, as opposed to the first one.)

Another way to solve the above problems is to directly calculate the number of ways in which the desired event occurs. For example, in part (**a**), you find out the number of outcomes with 1 six (this is $\binom{4}{1}\times 5^3$), the number of outcomes with 2 sixes ($=\binom{4}{2}\times 5^2$), the number of outcomes with 3 sixes ($=\binom{4}{3}\times 5$), and the number of outcomes with 4 sixes ($= 1$). Now add these up.
You will find that doing this for part (**b**) is not as easy!