University of Illinois | Problem Set 1 | ECE 313 Fall 1998

Assigned : Wednesday, August 26, 1998
Due : Wednesday, September 2, 1998
Reading : Ross, Chapter 1.1–1.4, Chapter 2.1–2.5
About the problems : Most of these problems are based on material covered in the *prerequisite* (only calculus) to this course. The problems are meant as a self-diagnosis aid; this material underlies most of the mathematical development in this course.

1. In this problem all angles are expressed in **radians**.
   (a) In this part you have to find the limit of $\dfrac{1}{\sin^2 x} - \dfrac{1}{\tan^2 x}$ as $x \to 0$. Use your calculator to evaluate this function for *small* values of x, say $x = 0.1, 0.01, 0.001, \ldots$. Does this function seem to be approaching a limit? If so, what do you think it is? Now, use what you have learnt about limits in calculus to find $\lim_{x\to 0}\left(\dfrac{1}{\sin^2 x} - \dfrac{1}{\tan^2 x}\right)$ analytically.
   (b) Repeat the above procedure for the function $f(x) = \dfrac{1}{\sin^2 x} - \dfrac{1}{x^2}$. (Hint: the answer is not $0, 1$ or ∞.)

2. Let $\dfrac{d}{dx}f(x) = g(x)$, $-\infty < x < \infty$. Which of the following statements is true for all x? (Here, C denotes an arbitrary constant.)
   (i) $\frac{d}{dx}f(-x) = -g(-x)$ (ii) $\frac{d}{dx}f(x^2/2) = xg(x^2/2)$ (iii) $\frac{d}{dx}\exp\left(f(x^2)\right) = g(x^2)\exp\left(f(x^2)\right)$
   (iv) $\int g(-x)\,dx = f(-x) + C$ (v) $\int g(x^2/2)\,dx = \frac{f(x^2/2)}{x} + C$ (vi) $\int \frac{g(x)}{f(x)}\,dx = \ln(f(x)) + C$

3. Find the values of the following **definite** integrals:
   (a) $\displaystyle\int_{-1}^{1} |x|\,dx$ (b) $\displaystyle\int_0^1 x(1-x^2)^{11}\,dx$.
   (c) $\displaystyle\int_{-\infty}^{\infty} x\exp(-x^2/2)\,dx$. (Hint: Plotting the integrand might give you some clues.)

4. Does there exist a function $f(x)$ satisfying the following two conditions : **(i)** $f(x) \geq 0$ for all real numbers in the range $a \leq x \leq b$, and **(ii)** $\displaystyle\int_a^b f(x)\,dx < 0$? (For example, does this hold for the integrals in parts (a) and (b) of the previous question?)

5. **[15 pts.]** Evaluate the following **definite** two-dimensional integrals over the specified domains of integration:
   (a) $f(x,y) = \min(x,y)$, over the region $\{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$.
   (The "min" or minimum function is defined by $\min(x,y) = x$ if $x \leq y$; $\min(x,y) = y$ if $y \leq x$.)
   (b) $f(x,y) = (x^2+y^2)^{-2}$, over the region $\{(x,y) : x^2+y^2 > 1\}$.

(c) $f(x,y) = \begin{cases} x+y, & 0 < x < y < 1 \\ 0, & \text{elsewhere} \end{cases}$, over the region $\{(x,y) : 0 < x+y < 1\}$.

6. **Super-Calvinball:** Calvin and Hobbes start running towards each other from opposite ends of a 100 meter long field, each running at a speed of $1m/sec$ (yeah, we're not talking world record speeds here). As they start, Calvin throws a baseball he's holding to Hobbes, who catches it in midstride and throws it back without wasting a second. Calvin, not to be outdone, catches the ball (in midstride, and without wasting any time either!) and throws it back, ad nauseum. They both throw the ball at a speed of $2m/sec$. If they continue this process till they collide at the center of the field (where they sit dazed for a minute or two before resuming Super-Calvinball), write an expression for how many meters the ball has travelled. Do you recognize this series? What does the expression simplify to? (What does this tell you about the relationship between the length of a question and its answer?!)

7. **Extra Credit [10 pts. for each part]:** (These were two problems in gambling that opened the doors to probability in the 17^{th} century.)

   (a) A popular game in gambling houses in 17^{th} century France led to the following question: What is the probability of rolling *at least one* "six" in four rolls of a 6-faced die?

   (b) A follow-up (albeit more intricate) game involved the following question: What is the probability of rolling *at least one pair* of "sixes" in twenty-four rolls of a pair of dice? (What could be the reason that the old-time gamblers considered **twenty-four** rolls of a pair of dice as a sequel to the first game?)

8. **Extra Credit:** What do you estimate the probability of your getting an "A" in this course? Explain with suitable references. (In technical terms, this probability would be called an *a priori* probability.)