Assigned: Wednesday, November 19, 1997
Due: Wednesday, November 26, 1997
Reading: Ross, Chapters 4 and 5
Suggested Noncredit Exercises: Ross, pp. 173-184: 20-23, 25, 27-29,34-38; pp. 184-188: 6, 7, 9, 20; pp. 232-237: 7-9, 12, 13, 15-19, 21,26, 29,30-34; pp. 237-241: 2, 3, 8, 12-22, 26, 29.
Noncredit Exercises: Those using the 4th edition should try Ross pp. 177-182: 6, 7, 9, 20; pp. 182-196: 20-23, 25, 27-29, 34-38; pp. 239-243: 2, 3, 8, 12-22, 26, 29; pp. 243-248: 7-9, 12, 13, 15-19, 21,26, 29,30-34.

Problems:

1. **Do not consult** a table of integrals, or a calculus book, or Mathematica/MATLAB, or a caclulator/laptop etc in doing this problem.

(a) exp(–ln 2) equals **(i)** 2, **(ii)** –2, **(iii)** 1/2, **(iv)** –1/2, **(v)** ln 2

(b) $\int \exp(-u)\,du$ equals **(i)** $\frac{\exp(u)}{-u}$, **(ii)** $\frac{\exp(-u)}{u}$, **(iii)** $\frac{\exp(u)}{u}$, **(iv)** $\frac{\exp(-u)}{-u}$, **(v)** $-\exp(-u)$

(c) $\int \exp(-|u|)\,du$ equals **(i)** $\frac{\exp(-|u|)}{-|u|}$, **(ii)** $\frac{\exp(-|u|)}{u}$, **(iii)** $\frac{\exp(-|u|)}{-u}$, **(iv)** $\frac{\exp(u)}{-u}$, **(v)** ∞

2. In Problem 7 of Problem Set 10, we proved that 1 = 0 from which it readily follows (on multiplying both sides by 2 or by π) that 2 = 0 or π = 0. Here, we give a different proof of this latter result. Where is the error (if any) in the proof?

Let $I = \int_{-1}^{1} \frac{2}{1+x^2}\,dx$. Since $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$ and $\arctan(\pm 1) = \pm\pi/4$, I equals π.

Or does it? Let us make the substitution t = 1/x in the integral. Then, $dx = -\frac{1}{t^2}\,dt$, the limits are still –1 and +1, and the integrand becomes $\frac{2}{1+1/t^2}$. Thus, after simplification we have that $I = \int_{-1}^{1} \frac{2}{1+x^2}\,dx = \int_{-1}^{1} \frac{-2}{1+t^2}\,dt$. Since the value of a definite integral does not depend on which letter denotes the variable in the integrand, the second integral equals –I. It follows that 2I = 0 from which we conclude that either 2 = 0, or I = 0. Since we had found earlier that I = π, we have shown that either 2 = 0 or π = 0.

3. **X** is uniformly distributed on [–1, +1].

(a) If $\mathbf{Y} = \mathbf{X}^2$, what are the mean and variance of **Y**?

(b) If **Z** = g(**X**) where $g(u) = \begin{cases} u^2, & u \geq 0, \\ -u^2, & u < 0, \end{cases}$ use LOTUS (or the EZ method) to find E[**Z**]

4. Participants in a lottery buy tickets for $1 each. Each ticket bears one of the numbers 1 to 32. One of these 32 numbers is drawn at random, and all holders of tickets with the winning number are paid $29 per ticket (i.e. the ticket price plus winnings of $28).

(a) If you buy a ticket numbered 7, what is the probability that you win? If you buy a ticket for each drawing of the lottery, what is the **average** number of drawings before you win? Is this average number calculation valid only if you buy a ticket numbered 7 each time, or can you vary your choice with each drawing?

(b) What is your average win (or loss) per drawing?

(c) If you buy 32 tickets numbered 1 through 32, you will always hold a winning ticket. But, how much money do you lose on each drawing? What is your loss **per ticket?**

(d) In response to complaints that the lottery payoffs are too low, the administrator claims that on average, number 7 (say) wins at least once in 22 drawings while he is paying out $28 on winning tickets. Thus, the lottery is actually biased in the player's favor! He backs up his claim by offering the following even-money bet: if 7 **wins** in the next 22 drawings, you will pay him, while if 7 **doesn't win** at least once in the next 22 drawings, he will pay you. You test the administrator's claim via many such bets, and find that he is right! On half of these bets, 7 does not occur in the next 22 drawings and you collect, but on the other half of your bets, 7 does occur in the next 22 drawings, and you have to pay up. But, is the bet *perfectly* fair (in the sense that the expected win/loss per bet is 0)? Does it prove the administrator's claim that 7 wins at least once (on average) in 22 drawings? If you agree with the administrator, how do you reconcile this with part (a) where you found (I hope!) that 7 wins once in 32 drawings (on average?)

5. A newsboy purchases H newspapers for c_2 cents each and sells them for c_3 cents each. He can return unsold papers to the publisher for c_1 cents each. Note that $c_1 < c_2 < c_3$. The daily demand $\mathbf{X}$ for papers is a integer-valued random variable with pmf $p_{\mathbf{X}}(u)$.

(a) What is the probability that he sells all H newsapers? Express your answer in terms of $F_{\mathbf{X}}(u)$.

(b) Let $\mathbf{Z}$ denote the daily profit (in cents) that the newsboy makes. Write an expression for $\mathbf{Z}$ in terms of $\mathbf{X}$ and H

(c) Write an expression for his average daily profit. Your answer will depend on H, so call the expression for the **average** daily profit the function g(H).

(d) The newsboy has been buying H papers for some months and making an average profit g(H) each day. One day, he decides to buy one extra paper. What is the probability that he can sell this extra paper? Show that he makes an average **additional** profit of $(c_3-c_2) - (c_3-c_1)F_{\mathbf{X}}(H)$ from the extra paper. Call this A(H).

(e) Show that the average **additional** profit $A(H) = (c_3-c_2) - (c_3-c_1)F_{\mathbf{X}}(H)$ satisfies

$$\ldots A(H-1) \geq A(H) \geq A(H+1) \ldots ,$$

that is, on average, each extra newspaper brings in smaller extra profit that the previous one. This is called the law of diminishing returns.

(f) Show that for sufficiently large H, A(H) is negative so that the newsboy loses money by buying too many extra papers.

(g) How many papers should he purchase to maximize his average profit?

6. We return to Problem 1 of Problem Set #10. Suppose that the owner makes a gross profit of $0.64 for each gallon of gasoline sold. Let $\mathbf{Y}$ denote the amount of gasoline sold per week.

(a) How is $\mathbf{Y}$ related to $\mathbf{X}$, the weekly demand for gasoline? (Hint: she cannot sell more gasoline each week than the tank can hold!)

(b) What is her **average** weekly gross profit?

(c) Now suppose that the owner pays $20C as weekly rent on a tank of capacity C. What is her **average** weekly **net** profit and what value of C maximizes her average weekly net profit?