Assigned: Wednesday, November 12, 1997
Due: **Friday, November 21,** 1997
Reading: Ross, Chapters 4 and 5
Reminder: Hour Exam II is scheduled for Monday November 17 from 7 p.m. to 8 p.m. in Room 228, Natural History Building

Suggested Noncredit Exercises: Ross, pp. 173-184: 17-19, 40-43, 49-56 70, 73; pp. 184-188: 5, 25-28;
pp. 232-237: 30, 31, 35-39; pp. 237-241: 14, 28, 29.

Noncredit Exercises: Those using the 4th edition should try Ross pp. 177-182: 5, 25-28; pp. 182-196: 17-19, 40-43, 49-56, 70, 73; pp. 239-243: 14, 28, 29; pp. 243-248: 30, 31, 35-39

Problems:

1. The weekly demand (measured in thousands of gallons) for gasoline at a rural gas station is a random variable **X** with probability density function

$$f_{\mathbf{X}}(u) = \begin{cases} 5(1-u)^4, & 0 < u < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Let C (in thousands of gallons) denote the capacity of the tank (which is re-filled weekly.)

(a) If C = 0.5 and **X** happens to have value 0.68 one particular week, can the gas station satisfy the demand that week? That is, can the gas station supply gasoline to all those who demand it that week?

(b) If C = 0.5 and **X** happens to have value 0.43 some other week, can the gas station satisfy the demand during this other week? That is, can the gas station supply gasoline to all those who demand it that week?

(c) If C = 0.5, what is the *probability* that the weekly demand for gasoline can be satisfied? Note that if your answer is (say) 0.666…, then, in the long run, the gas station can supply the weekly demand two weeks out of three.

(d) What is the minimum value of C required to ensure that the probability that the demand exceeds the supply is no larger than 10^{-5} ?

2. **X** is a geometric random variable with parameter 1/2, and $\mathbf{Y} = \sin(\pi\mathbf{X}/2)$.
Is **Y** a continuous random variable or a discrete random variable or a mixed random variable?
If you think that **Y** is a continuous random variable, find the pdf of **Y**.
If you think that **Y** is a discrete random variable, find the pmf of **Y**.
If you think that **Y** is a mixed random variable, find the CDF of **Y**.

3. The random variable **X** has probability density function $f_{\mathbf{X}}(u) = \begin{cases} 2(1-u), & 0 \le u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$

Let $\mathbf{Y} = (1 - \mathbf{X})^2$.

(a) What is the minimum value of **Y**? Call this α.
What is the maximum value of **Y**? Call this β.
What do you think are the values of $P\{\mathbf{Y} \le \alpha - 1\}$ and $P\{\mathbf{Y} > 2\beta\}$?

(b) What is the CDF $F_{\mathbf{Y}}(v)$ of the random variable **Y**?
Be sure to specify the value of $F_{\mathbf{Y}}(v)$ for all v, $-\infty < v < \infty$.

(c) Show that the CDF $F_{\mathbf{Y}}(v)$ that you found in part (b) is a nondecreasing continuous function. Is $F_{\mathbf{Y}}(v)$ differentiable at α? at β?

(d) From the definition of the CDF $F_{\mathbf{Y}}(v)$, we know that $P\{\mathbf{Y} \le \alpha - 1\} = F_{\mathbf{Y}}(\alpha - 1)$ and $P\{\mathbf{Y} > 2\beta\} = 1 - F_{\mathbf{Y}}(2\beta)$. Does substituting $v = \alpha - 1$ and $v = 2\beta$ in the CDF $F_{\mathbf{Y}}(v)$ that you found in part (b) give the same values for $P\{\mathbf{Y} \le \alpha - 1\}$ and $P\{\mathbf{Y} > 2\beta\}$ that you stated in part (a)? If the values are different, which ones are the correct values? Explain your choices of correct answer (and why the other possible answers are wrong) in detail.

4. [Read Example 3d on pp. 203-204 (pp. 209-210 for 4th editioners) first.]
Let the (straight) line segment ACB be a diameter of a circle of unit radius and center C. Consider an arc AD of the circle where the length **X** of the *arc* (measured clockwise around the circle) is a random variable uniformly distributed on $[0,2\pi)$. Now consider the "random chord" AD.

(a) Find the probability that the length **L** of the random chord is greater than the side of the equilateral triangle inscribed in the circle.

(b) Express **L** as a function of the random variable **X**, and find the probability density function for **L**.

5. The radius of a sphere is a random variable **R** with pdf $f_{\mathbf{R}}(\rho) = \begin{cases} 3\rho^2, & 0 < \rho < 1, \\ 0 & \text{elsewhere.} \end{cases}$

(a) Find the CDF $F_{\mathbf{V}}(\alpha)$ and pdf $f_{\mathbf{V}}(\alpha)$ of **V**, the volume of the sphere.

(b) If the sphere is made of metal and carries an electrical charge of Q coulombs, what is the CDF $F_{\mathbf{S}}(x)$ and the pdf $f_{\mathbf{S}}(x)$ of the surface charge density **S** on the sphere ?

6. The voltage V across and the current I through a semiconductor diode are related as $I = e^V - 1$. If a random voltage **X** with a Laplacian distribution with parameter 1 (see Ross, p. 219 (p. 227 for 4th edition users)) is applied to this diode, find the pdf of the resulting current **Y** through the diode. Draw a neat sketch of the pdf of **Y**.

7. The following argument proves that 0 = 1, or does it?. Is there an error in the argument? If so, where is the error? If the argument is correct, why doesn't your calculator agree?

The "integration by parts" formula states that $\int u\,dv = uv - \int v\,du$. We apply this to the

computation of $\int \frac{1}{x}\,dx$ where we take $u = \frac{1}{x}$, $v = x$, so that $du = \left(\frac{-1}{x^2}\right)dx$, and $dv = dx$.

Plugging and chugging, we thus have $\int \frac{1}{x}\,dx = \frac{1}{x}\,x - \int x\left(\frac{-1}{x^2}\right)dx = 1 + \int \frac{1}{x}\,dx$.

Subtracting $\int \frac{1}{x}\,dx$ from both sides, we conclude that 0 = 1.