Assigned: Wednesday, October 1, 1997
Due: Wednesday, October 8, 1997
Reminders: **No class on Friday October 3.**
Hour Exam I is on Tuesday October 14, 7 pm to 8 pm, in 228 Natural History Building. For coverage of material on the exam, consult the ECE 313 home page at http://tesla.csl.uiuc.edu/~sarwate/
Reading: Ross, Chapter 3
Noncredit Exercises: (Do not turn these in) Ross pp.104-117: 53, 58, 59, 62, 63, 70-74, 78, 81
Those with the **4th** edition should try: Ross, pp. 112-125: 50, 52, 53, 57, 58, 64-69, 73, 76.
Problems:

1. A professor wishes to determine whether a student is excellent, good, or average which he defines as hypotheses H_0, H_1, and H_2 respectively. Let A, B, C, and D respectively denote the events that a student's performance on an exam is excellent, good, average, or poor. From previous experience, the professor knows that $P_0(A) = 0.75$, $P_0(B) = 0.15$, $P_0(C) = 0.08$, and $P_0(D) = 0.02$, that is, an excellent student has 75% chance of having an excellent performance on an exam, 15% chance of having a good performance, 8% chance of having an average performance, and 2% chance of having a poor performance. Similarly, $P_1(A) = 0.15$, $P_1(B) = 0.6$, $P_1(C) = 0.15$, $P_1(D) = 0.1$; and $P_2(A) = 0.05$, $P_2(B) = 0.1$, $P_2(C) = 0.65$, $P_2(D) = 0.2$.

(a) For each of the four possible results of the student's exam (excellent/good/average/poor performance), find the professor's maximum-likelihood decision as to whether the student is excellent, or good or average.

(b) What is the probability that an excellent student is mistakenly labeled as good by the professor? What is the probability that an excellent student is mistakenly labeled as average by the professor? What is the probability that an average student is classified as being above average (that is, as good or excellent)?

(c) If 20% of the students are excellent, 55% are good, and 25% are average (that is, $P(H_0) = 0.2$, $P(H_1) = 0.55$, and $P(H_2) = 0.25$, what is the probability that the professor's maximum-likelihood decision rule mis-classifies students?

(d) What is the Bayes' decision rule corresponding to these probabilities and what is the probability that the Bayes' decision rule mis-classifies students?

(e) At the Lake Wobegon campus of the University of Illinois, 95% of students are excellent and 5% are good (and thus they are all above average!) What is Bayes' decision rule in this case? That is, what does the Bayesian professor decide about a student based on the four possible results of the student's exam?

(f) For what range of values of $P(H_0)$ does the Bayes' decision rule always decide that H_0 is true? Note that the decision rule should be applicable regardless of the values of $P(H_1)$ and $P(H_2)$ (subject, of course to the constraints that $P(H_0) \geq 0$, $P(H_1) \geq 0$, $P(H_2) \geq 0$, and that $P(H_0) + P(H_1) + P(H_2) = 1$)

2. Urns A, B, and C contain one black ball each, and in addition, respectively one, two, and three white balls. An urn is chosen and a ball drawn from the urn.

(a) What is the maximum-likelihood rule for deciding which urn has been chosen based on observation of the color of the ball drawn?

(b) Now suppose the maximum-likelihood rule is being used, but the urns are chosen with probabilities $P(A) = 1/2$, $P(B) = P(C) = 1/4$. Find $P(E)$, the probability of making a wrong decision.

(c) Find the Bayes' rule for the prior probabilities of part (b) and show that its error probability is smaller than that of the maximum-likelihood decision rule.

(d) Now suppose that the prior probabilities are $P(A) = 6/23$, $P(B) = 9/23$, and $P(C) = 8/23$. Find the Bayes decision rule and show that $P(E) = 14/23$.

(e) Now consider the following *randomized* decision rule: If you see a black ball, decide in favor of A, B, C with probabilities 18/23, 5/23, and 0 respectively, while if you see a white ball, decide in favor of A, B, C with probabilities 0, 11/23, and 12/23 respectively. Write an expression for P(E) in terms of P(A), P(B), and P(C).

3. Let A and B denote events defined on a sample space Ω. The probability that at least one of the two events occurred is 0.8, and the probability that at least one of the two events did not occur is 0.7. If A and B are known to be independent events such that P(A) > P(B), find P(A) and P(B). Is the answer unique ?

4. Ross, p. 111, #51 (**#48** on page **119** for those using the **4th** edition).

5. The Senate of a certain country has 100 members consisting of 43 Conservative Republicans, 21 Conservative Democrats, 12 Liberal Republicans, and 24 Liberal Democrats. Before each vote, the groups caucus separately. Each group decides independently of the other groups whether to support or oppose the motion. *All* members of the group then vote in accordance with the caucus decision. For those who think that this is the way politics works, I have this beautiful skyscraper on Wacker Drive in Chicago that I am willing to sell to you at a bargain price…

(a) Let A, B, C, and D respectively denote the events that the four groups vote for a spending plan that will lead to a balanced budget in seven years. If the probabilities of these independent events are P(A) = 0.9, P(B) = 0.6 P(C) = 0.5 and P(D) = 0.2, what is the probability that the bill passes ?

(b) The President vetoes the bill. Let E, F, G, and H respectively denote the independent events that the four groups support the motion to override the veto. If these events have probabilities P(E) = 0.99, P(F) = 0.4, P(G) = 0.6, and P(H) = 0.1, what is the probability that the motion to override the veto passes ?
Political innocents are reminded that a simple majority (51 or more votes) is required to pass a bill, and a two-thirds majority (67 or more votes) to override a veto.

6. Two of the eleven letters in a road sign that reads CHATTANOOGA have fallen down. Assume that each pair of letters is equally likely to have fallen down. A drunk randomly puts the fallen letters back into the two empty slots, possibly interchanging the positions of the letters, and possibly putting the letters back upside down. Note that an upside-down letter is one that appears right side up to those standing on their heads, and be aware that four of the letters (which ones?) in CHATTANOOGA look the same even when they are upside down. Assume that the events

A = {both replaced letters are in their correct slots},
B = {leftmost replaced letter is correct side up}
and C = {rightmost replaced letter is correct side up}

are independent events and that each has probability 1/2.

(a) What is the probability that the sign still **seems to read** CHATTANOOGA?

(b) What is the probability that all the letters **seem** to be correct side up but the sign does not read CHATTANOOGA?

(c) What is the probability that only one letter **seems** to be upside down?

(d) What is the probability that two letters **seem** to be upside down?

(e) Given that all the letters appear to be right side up, what is the probability that **at least one** vowel fell down?

(f) A designated driver observes the restored sign and makes a decision as to the most likely letters to have fallen down. The driver **knows** that the sign is supposed to read CHATTANOOGA. What is the probability that the driver can **correctly** identify (without any possibility of making a mistake) which letters fell down?

(g) What is the Bayes' rule decision if the sign reads CHATTANOOGA? (Hint: the answer is not unique, so the driver chooses one of the many possibilities) What is the (conditional) probability that this decision is correct (given that the sign reads CHATTANOOGA, of course)?