

# HW10 Solutions

ECE 310: Digital Signal Processing, Spring 2026

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## Solution:

### 1. FFT:

The decimation-in-time algorithm rearranges the input in bit-reversed order:  $x[0], x[2], x[1], x[3]$ . For  $x = \{2, 0, 1, -1\}$ :

- **Stage 1** (2-point butterflies):
  - Upper:  $X_1(0) = x[0] + x[2] = 2 + 1 = 3$
  - Upper:  $X_1(1) = x[0] - x[2] = 2 - 1 = 1$
  - Lower:  $X_2(0) = x[1] + x[3] = 0 + (-1) = -1$
  - Lower:  $X_2(1) = x[1] - x[3] = 0 - (-1) = 1$
- **Stage 2** (4-point butterflies with twiddles  $W_4^0 = 1, W_4^1 = -j$ ):
  - $X(0) = X_1(0) + W_4^0 X_2(0) = 3 + (1)(-1) = 2$
  - $X(1) = X_1(1) + W_4^1 X_2(1) = 1 + (-j)(1) = 1 - j$
  - $X(2) = X_1(0) - W_4^0 X_2(0) = 3 - (1)(-1) = 4$
  - $X(3) = X_1(1) - W_4^1 X_2(1) = 1 - (-j)(1) = 1 + j$

**FFT Output:**  $X = \{2, 1 - j, 4, 1 + j\}$

### 2. Standard DFT Comparison:

Using  $X[k] = \sum_{n=0}^3 x[n]e^{-j\frac{2\pi}{4}nk}$ :

- $X[0] = 2 + 0 + 1 - 1 = 2$
- $X[1] = 2(1) + 0(-j) + 1(-1) - 1(j) = 1 - j$
- $X[2] = 2(1) + 0(-1) + 1(1) - 1(-1) = 4$
- $X[3] = 2(1) + 0(j) + 1(-1) - 1(-j) = 1 + j$

**DFT Output:**  $X = \{2, 1 - j, 4, 1 + j\}$

**Grading:** 30 points

- **10 pts:** Correct 4-point FFT flow graph (bit-reversal and twiddles).
- **10 pts:** Correct computation of the FFT output values.
- **10 pts:** Correct verification using standard DFT formula.

**Solution:****1.  $i$  and  $j$ :**

In the final stage (Stage 6) of a 64-point FFT, the butterfly span is  $N/2 = 32$ . In the penultimate stage (Stage 5), the span is  $N/4 = 16$ .

- The butterfly involving  $X[37]$  and  $X[i]$  belongs to the final stage (Stage 6). The span is 32. Since 37 is the lower node,  $X[37 - 32] = X[5]$  is the upper node of this butterfly (as it connects to a node further down).
- The butterfly involving  $X[i]$  and  $X[j]$  belongs to the penultimate stage (Stage 5). We know they are in a butterfly together because of the bottom left butterfly in stage 5. Therefore, the span is 16. Since  $X[5]$  is the upper node, its corresponding lower node is  $j = i + 16 = 21$ .

**Indices:**  $i = 5, j = 21$ .

**2.  $a, b, c, d$ :**

Twiddle factors  $W_{64}^k$  are applied to the lower branches (before the split) of the butterflies at stage 6 and split to [16, 31] and [48, 64] at stage 5. (To get a better understanding of this please look at the DIT FFT of 8 below. Most notably look how the twiddles split in two as you go back from the final stage backwards.)

- **Coefficient  $a$ :** This is the straight-through top half of the final stage  $X[21]$ . Thus,  $a = 1$ .
- **Coefficient  $b$ :** This is the straight-through lower branch of the Stage 6 butterfly  $X[37]$ . This means the  $k$  in the twiddle factor is  $37 - 32 = 5$ . We also must account for the -1 that all lower branches get since it is a straight-through branch, meaning  $b = -W_{64}^5$ .
- **Coefficient  $c$ :** We can apply the same logic to  $c$ . It is also a lower straight-through branch. Since we know  $X[j]$  is 32 away from the output where  $c$  goes to, we know that output is  $X[53]$ . Using the same logic for  $b$ , we get the  $k$  in the twiddle to be  $53 - 32 = 21$ , meaning  $c = -W_{64}^{21}$ .
- **Coefficient  $d$ :** Stage 5 is a bit trickier. We can notice the straight-through index of  $d$  will be  $X[53]$ . This means we will have a twiddle factor here, due to "Twiddle factors split to [16, 31] and [48, 64] at stage 5", stated above. The formula to get the twiddle factor here is halving the stage 6 twiddle factor then flooring it. stage 6 twiddle was 21, so  $\lfloor 21/2 \rfloor = 10$ . We should not forget the negative which happens in straight-through branches with twiddle factors. So  $d = -W_{64}^{10}$ . (This explanation is a bit hand wavy and hard to explain through text, but if having trouble stare at the 8 length diagram below and relate it to the 64 one here.)

**Alternative Explanation:** Let's say  $X[k]$  is the result of combining length-32 DFTs  $A[k]$  (even) and  $B[k]$  (odd). From the previous parts we can see that the two nodes computed by stage 5 are  $B[5]$  and  $B[21]$ . From looking at the diagram (and since  $21 > 15$ ), the equation of interest is  $B[k+16] = C[k] - W_{32}^k D[k]$ . Plugging in  $k=5$  gives  $B[21] = C[5] - W_{32}^5 D[5]$ . So  $d = -W_{32}^5 = -W_{64}^{10}$ .

**Coefficients:**  $a = 1, b = -W_{64}^5, c = -W_{64}^{21}, d = -W_{32}^5 = -W_{64}^{10}$ .

**Grading:** 30 points

- **10 pts:** Correct indices ( $i = 21, j = 21$ ) based on Stage 5/6 spans.
- **20 pts:** Correct coefficients ( $a = 1, b = W_{64}^5, c = -1, d = W_{64}^{10}$ ).

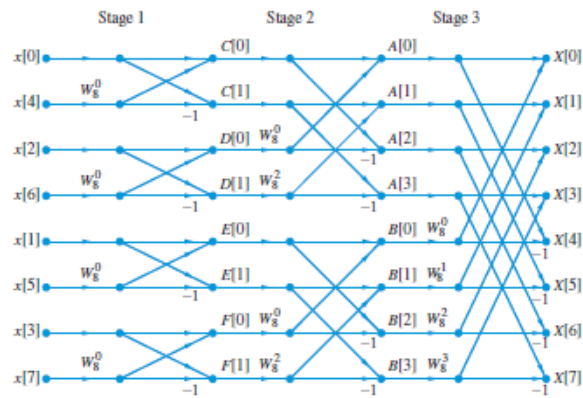


Figure 8.6 Flow graph of 8-point decimation-in-time FFT algorithm using the butterfly computation shown in Figure 8.4. The trivial twiddle factor  $W_8^0 = 1$  is shown for the sake of generality.

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### Solution:

(a) **Linear Convolution**  $x[n] * h[n]$ :

The sequence lengths are  $L = 5$  and  $M = 3$ . The resulting length is  $L + M - 1 = 7$ . Using the sliding window:

- $y[0] = 1(-1) = -1$
- $y[1] = 2(-1) + 1(0) = -2$
- $y[2] = 3(-1) + 2(0) + 1(1) = -2$
- $y[3] = 2(-1) + 3(0) + 2(1) = 0$
- $y[4] = 1(-1) + 2(0) + 3(1) = 2$
- $y[5] = 1(0) + 2(1) = 2$
- $y[6] = 1(1) = 1$

OR

(a) **Linear Convolution via Matrix Multiplication:**

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \\ y[6] \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -2 \\ 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

**Result:**  $y[n] = \{-1, -2, -2, 0, 2, 2, 1\}$  starting at  $n = 0$

**(b) 5-point Circular Convolution  $x[n] \otimes_5 h_{zp}[n]$ :**

Since  $N = 5$ , the linear convolution  $y[n]$  will experience "aliasing" where  $y[n] + y[n + 5]$ .

- $c[0] = y[0] + y[5] = -1 + 2 = 1$
- $c[1] = y[1] + y[6] = -2 + 1 = -1$
- $c[2] = y[2] = -2$
- $c[3] = y[3] = 0$
- $c[4] = y[4] = 2$

**OR**

**(b) 5-point Circular Convolution via Circulant Matrix:**

$$\begin{bmatrix} c[0] \\ c[1] \\ c[2] \\ c[3] \\ c[4] \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 + 2 \\ -2 + 1 \\ 1 - 3 \\ 2 - 2 \\ 3 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \\ 2 \end{bmatrix}$$

**Result:**  $\{1, -1, -2, 0, 2\}$  starting at  $n = 0$

**(c) Smallest  $N$  for Circular = Linear:**

For the  $N$ -point circular convolution to equal the linear convolution,  $N$  must be at least the length of the linear convolution.  $N \geq L + M - 1 \implies N \geq 5 + 3 - 1 = 7$ . **Minimum Value:**  $N = 7$ .

**Grading:** 20 points

- **5 pts:** Correct linear convolution (Part a).
- **10 pts:** Correct circular convolution (Part b).
- **5 pts:** Correct minimum  $N$  and reasoning (Part c).

**Solution:****(a) Length of the sequence  $y[n]$ :**

The length of the linear convolution of two sequences with lengths  $L$  and  $M$  is given by  $N = L + M - 1$ .

- Length of  $x[n]$  ( $L$ ) = 94 (since  $n = 0$  to 93)
- Length of  $h[n]$  ( $M$ ) = 53 (since  $n = 0$  to 52)
- Length of  $y[n] = 94 + 53 - 1 = 146$

**Result:** 146

**(b) Zero-padding for the smallest radix-2 FFT:**

To use a radix-2 FFT for linear convolution, the FFT size  $N_{FFT}$  must satisfy two conditions:

1.  $N_{FFT} \geq L + M - 1$  to avoid time-domain aliasing.
2.  $N_{FFT} = 2^\nu$  where  $\nu$  is an integer (the radix-2 requirement).

From part (a), we know  $N_{FFT} \geq 146$ .

The powers of 2 are:  $2^7 = 128$  (too small) and  $2^8 = 256$ .

- The smallest power of 2 greater than or equal to 146 is **256**.
- Both sequences should be zero-padded to a total length of 256.
- For  $x[n]$ , add  $256 - 94 = 162$  zeros.
- For  $h[n]$ , add  $256 - 53 = 203$  zeros.

**Smallest radix-2 FFT size:** 256

**Grading:** 20 points

- **5 pts:** Correct calculation of  $y[n]$  length (146).
- **5 pts:** Correct identification of the power-of-2 requirement ( $N \geq 146 \implies 256$ ).
- **10 pts:** Correct explanation of zero-padding for both sequences.