HW5 Solutions

ECE 310: Digital Signal Processing, Spring 2026 Version: **1.0**

Due Date: 2/29 @ 11:59pm
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Solution:

(a) $x[n] = \{1, 0, \underset{\uparrow}{0}, 0, -1\}$

First, we re-express $x[n]$ as a sum of Kronecker deltas:

$$x[n] = \delta[n+2] - \delta[n-2]$$

We can then plug $x[n]$ into the definition of the DTFT to get

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} (\delta[n+2] - \delta[n-2])e^{-j\omega n} \\ &= e^{j2\omega} - e^{-j2\omega} \quad (*) \\ &= \boxed{2j\sin(2\omega)}. \end{aligned}$$

All (*) are acceptable solutions.

(b) $x[n] = u[n] - u[n-4]$

Taking a similar approach to part **(a)**:

$$\begin{aligned} x[n] &= \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] \\ \rightarrow X(\omega) &= \sum_{n=-\infty}^{\infty} (\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3])e^{-j\omega n} \\ &= 1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} \quad (*) \end{aligned}$$

Going further, we can express our result as the sum of two cosines:

$$\begin{aligned} X(\omega) &= (1 + e^{-j3\omega}) + (e^{-j\omega} + e^{-j2\omega}) \quad (*) \\ &= e^{-j\frac{3}{2}\omega}(e^{j\frac{3}{2}\omega} + e^{-j\frac{3}{2}\omega}) + e^{-j\frac{3}{2}\omega}(e^{j\frac{1}{2}\omega} + e^{-j\frac{1}{2}\omega}) \quad (*) \\ &= \boxed{2e^{-j\frac{3}{2}\omega}\left(\cos\left(\frac{1}{2}\omega\right) + \cos\left(\frac{3}{2}\omega\right)\right)} \end{aligned}$$

All (*) are acceptable solutions.

(c) $x[n] = \cos(\frac{\pi}{3}n + \frac{\pi}{4})$

We'll start by expressing $x[n]$ in terms of exponentials:

$$\begin{aligned}\cos\left(\frac{\pi}{3}n + \frac{\pi}{4}\right) &= \frac{1}{2}\left(e^{j\left(\frac{\pi}{3}n+\frac{\pi}{4}\right)} + e^{-j\left(\frac{\pi}{3}n+\frac{\pi}{4}\right)}\right) \\ &= \frac{1}{2}e^{j\frac{\pi}{4}}e^{j\frac{\pi}{3}n} + \frac{1}{2}e^{-j\frac{\pi}{4}}e^{-j\frac{\pi}{3}n}\end{aligned}$$

Using the DTFT pair given and the linearity of the DTFT, we have

$$\begin{aligned}X(\omega) &= \frac{1}{2}e^{j\frac{\pi}{4}}\left(2\pi\delta\left(\omega - \frac{\pi}{3}\right)\right) + \frac{1}{2}e^{-j\frac{\pi}{4}}\left(2\pi\delta\left(\omega + \frac{\pi}{3}\right)\right) \quad (*) \\ &= \boxed{\pi e^{j\frac{\pi}{4}}\delta\left(\omega - \frac{\pi}{3}\right) + \pi e^{-j\frac{\pi}{4}}\delta\left(\omega + \frac{\pi}{3}\right)}.\end{aligned}$$

All (*) are acceptable solutions.

(d) $x[n] = \alpha^n e^{j\omega_0 n}u[n]$

Let $g[n] = e^{j\omega_0 n}$ and $h[n] = \alpha^n u[n]$. From the DTFT pair given, we know that the DTFT of $g[n]$ is $G(\omega) = 2\pi\delta(\omega - \omega_0)$. To find $H(\omega)$, however, we must refer to the definition of the DTFT:

$$\begin{aligned}H(\omega) &= \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n\end{aligned}$$

Since $|\alpha| < 1$, $|\alpha e^{-j\omega}| < 1$ for all ω. As such, we can apply the formula for the sum of an infinite geometric series,

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r},$$

giving us

$$H(\omega) = \frac{1}{1 - \alpha e^{-j\omega}}.$$

To determine the DTFT of $x[n] = g[n]h[n]$, we invoke the windowing property of the DTFT:

$$\begin{aligned}X(\omega) &= \frac{1}{2\pi}(G(\omega) * H(\omega)) \\ &= \frac{1}{2\pi}\left(2\pi\delta(\omega - \omega_0) * \frac{1}{1 - \alpha e^{-j\omega}}\right) \\ &= \boxed{\frac{1}{1 - \alpha e^{-j(\omega - \omega_0)}}}\end{aligned}$$

Grading: *60 points, 15 points each*

- For each of (a), (b), (c), and (d):
  - +15: correct answer with work shown (if incorrect answer but approach is correct, -3 points for each minor math mistake)
  - +6: correct answer without work shown OR correct answer using a DTFT table pair which was not given
  - +0: empty or invalid

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Solution:

(a) $X_d(0)$

$$\begin{aligned} X_d(0) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(0)n} \\ &= \sum_{n=-2}^{2} x[n] \\ &= \boxed{3} \end{aligned}$$

(b) $X_d(\pi)$

$$X_d(\pi) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\pi n}$$

From inspecting this expression, we can see that we're modifying $x[n]$ to be negative at all odd indices of n while keeping the signal at even indices the same. Then, we're taking the sum of this modified signal over all n. We can write this as

$$X_d(\pi) = \sum_{n=-2}^{2} f(n),$$

where

$$f(n) = \begin{cases} -x[n], & n \text{ odd} \\ x[n], & n \text{ even} \end{cases}$$

Therefore,

$$X_d(\pi) = \boxed{7}.$$

(c) $\int_{-\pi}^{\pi} X_d(\omega)d\omega$

If we recall the formula for the inverse DTFT,

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega,$$

we can see that setting $n = 0$ gives us

$$\begin{aligned} x[0] &= \frac{1}{2\pi}\int_{2\pi} X(\omega)d\omega \\ \rightarrow \int_{-\pi}^{\pi} X_d(\omega)d\omega &= 2\pi x[0] \\ &= \boxed{2\pi}. \end{aligned}$$

(d) $\int_{-\pi}^{\pi} |X_d(\omega)|^2 d\omega$

First, we should recognize that this expression looks very similar to Parseval's relation:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\omega)|^2 d\omega$$

This tells us that $\int_{-\pi}^{\pi} |X_d(\omega)|^2 d\omega$ equals the sum of the squared magnitudes of $x[n]$ over all n multiplied by 2π, such that:

$$\begin{aligned} \int_{-\pi}^{\pi} |X_d(\omega)|^2 d\omega &= 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 \\ &= 2\pi(4+1+1+1+4) \\ &= \boxed{22\pi} \end{aligned}$$

Grading: *40 points, 10 points each*

- For each of (a), (b), (c), and (d):
  - +10: correct answer without using closed form expression for $X_d(\omega)$
  - +4: correct answer, but used closed form expression for $X_d(\omega)$
  - +0: empty or invalid