HW Solutions
ECE 310: Digital Signal Processing, Spring 2026 Version: **1.0**

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Solution:

Z-transform of $x[n]$:

$X(z) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n = \frac{\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}, \quad$ ROC: $|z| > \frac{1}{2}$.

Z-transform of $h[n]$:

$H(z) = \sum_{n=-\infty}^{-1} (-2^n) z^{-n}$. Let $m = -n$:

$H(z) = -\sum_{m=1}^{\infty} \left(\frac{z}{2}\right)^m = -\frac{\frac{z}{2}}{1-\frac{z}{2}} = \frac{1}{1-2z^{-1}} \quad$ ROC: $|z| < 2$.

Now we can compute Z-transform of $y[n]$:

$$Y(z) = H(z)X(z) \tag{1}$$

$$= \frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} \quad \text{ROC: } \frac{1}{2} < |z| < 2 \tag{2}$$

$$= \frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1-2z^{-1}} \tag{3}$$

The region of convergence (ROC) of $Y(z)$ is $\frac{1}{2} < |z| < 2$ since it must at least contain the intersection of the original ROCs of $X(z)$ and $H(z)$. Now find the inverse Z-transform of individual terms:

- $\dfrac{1}{1-\frac{1}{2}z^{-1}}$ corresponds to $\left(\frac{1}{2}\right)^n u[n]$

- $\dfrac{1}{1-2z^{-1}}$ corresponds to $-2^n u[-n-1]$

Then final answer: $y[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{1}{3}2^n u[-n-1]$

Grading: *20 points*

- $+5$ for correct $H(z)$
- $+5$ for correct $X(z)$
- $+3$ for correct $Y(z)$, $+2$ for correct ROC of $Y(z)$
- $+5$ for correct $y[n]$

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Solution:

(a) $y[n] = (x[n])^2$.

Stable, because the square of any finite number will also be finite.

$$|x[n]| \leq M_x < \infty \longrightarrow |y[n]| = |x[n]|^2 \leq M_x^2 < \infty$$

(b) $y[n] = n(x[n])$.

Not stable, because as $n \longrightarrow \infty$, $y[n] \longrightarrow \infty$.

(c) $y[n] = (x * u)[n]$.

Not stable, consider the following bounded input which produces an unbounded output:

$$x[n] = 1 \; \forall \; n$$

$$y[n] = (x * u)[n] = \sum_{k=-\infty}^{\infty} 1u[n-k] = \sum_{k=-\infty}^{n} 1 = \infty$$

Another valid solution is to consider the Z-transform of the unit step function $u[n]$, which is $\frac{1}{1-z^{-1}}$. If a system's impulse response was the unit step, we can see from its Z-transform that there is a pole at 1, so the ROC is $|z| > 1$. Since the ROC does not contain the unit circle, the system is **not stable**.

Grading: *20 points*

- +6 pts correct part (a)
- +6 pts correct part (b)
- +8 pts correct part (c)

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Solution:

(a)
We are considering the following LCCDE:

$$y[n] = \frac{1}{6}y[n-1] + \frac{1}{6}y[n-2] + 2x[n] + x[n-1]$$

which has Z-transform

$$Y(z) = \frac{1}{6}z^{-1}Y(z) + \frac{1}{6}z^{-2}Y(z) + 2X(z) + z^{-1}X(z)$$

$$Y(z) - \frac{1}{6}z^{-1}Y(z) - \frac{1}{6}z^{-2}Y(z) = 2X(z) + z^{-1}X(z)$$

$$Y(z)\left(1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}\right) = X(z)\left(2 + z^{-1}\right)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{2 + z^{-1}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}$$

$$H(z) = 2\frac{1 + \frac{1}{2}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})}$$

The LCCDE represents a causal system because $y[n]$ only depends on current/past $x[n]$. Therefore the ROC is $|z| > \frac{1}{2}$.

(b)
By inspection of the transfer function $H(z)$:
Zeros: $z = -\frac{1}{2}, 0$
Poles: $z = \frac{1}{2}, -\frac{1}{3}$

(c)
To find $h[n]$, we need to perform partial fraction decomposition on $H(z)$

$$H(z) = 2\left(\frac{1 + \frac{1}{2}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})}\right) = 2\left(\frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 + \frac{1}{3}z^{-1}}\right)$$

$$A = \frac{1 + \frac{1}{2}(\frac{1}{2})^{-1}}{1 + \frac{1}{3}(\frac{1}{2})^{-1}} = \frac{1 + 1}{1 + \frac{2}{3}} = \frac{2}{\frac{5}{3}} = \frac{6}{5}$$

$$B = \frac{1 + \frac{1}{2}(-\frac{1}{3})^{-1}}{1 - \frac{1}{2}(-\frac{1}{3})^{-1}} = \frac{1 - \frac{3}{2}}{1 + \frac{3}{2}} = \frac{-\frac{1}{2}}{\frac{5}{2}} = -\frac{1}{5}$$

$$H(z) = 2\left(\frac{\frac{6}{5}}{1 - \frac{1}{2}z^{-1}} - \frac{\frac{1}{5}}{1 + \frac{1}{3}z^{-1}}\right)$$

$$h[n] = 2\left(\frac{6}{5}(\frac{1}{2})^n u[n] - \frac{1}{5}(-\frac{1}{3})^n u[n]\right)$$

(d)
Stable, because the ROC contains the unit circle.

(e)
To find the output $y[n]$ for the given input, we can take the Z-transform of $x[n]$, multiply by $H(z)$, then inverse Z-transform the result.

$$x[n] = 3\delta[n] + \delta[n-1]$$

$$X(z) = 3 + z^{-1} = 3(1 + \frac{1}{3}z^{-1})$$

$$Y(z) = H(z)X(z) = 2\left(\frac{1 + \frac{1}{2}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})}\right)3(1 + \frac{1}{3}z^{-1})$$

$$Y(z) = 6\left(\frac{(1 + \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})}\right)$$

$$Y(z) = 6\left(\frac{1 + \frac{1}{2}z^{-1}}{1 - \frac{1}{2}z^{-1}}\right)$$

$$Y(z) = 6\left(\frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{2}z^{-1}\frac{1}{1 - \frac{1}{2}z^{-1}}\right)$$

$$y[n] = 6\left((\frac{1}{2})^n u[n] + \frac{1}{2}(\frac{1}{2})^{n-1}u[n-1]\right)$$

$$y[n] = 6(\frac{1}{2})^n u[n] + 3(\frac{1}{2})^{n-1}u[n-1]$$

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Solution:

Z-transform of $h_1[n]$: $H_1(z) = -z + 2 - z^{-1}$

ROC: $0 < |z| < \infty$ (entire z-plane except possibly 0 and ∞). ROC includes the unit circle, hence $h_1[n]$ system is BIBO stable

Z-transform of $h_2[n]$:

$$h_2[n] = (n+1)u[n]$$
$$H_2(z) = \mathcal{Z}\{(n+1)u[n]\}$$
$$H_2(z) = \mathcal{Z}\{u[n]\} + \mathcal{Z}\{nu[n]\}$$
$$\mathcal{Z}\{u[n]\} = \frac{1}{1 - z^{-1}}, \qquad |z| > 1$$
$$\mathcal{Z}\{nu[n]\} = \frac{z^{-1}}{(1 - z^{-1})^2}, \qquad |z| > 1$$
$$H_2(z) = \frac{1}{1 - z^{-1}} + \frac{z^{-1}}{(1 - z^{-1})^2} = \frac{1}{(1 - z^{-1})^2}, \qquad |z| > 1$$

ROC does not include the unit circle, hence $h_2[n]$ system is not BIBO stable

Overall system:

$$H(z) = H_1(z)H_2(z)$$
$$H(z) = \left(-z + 2 - z^{-1}\right)\frac{1}{(1 - z^{-1})^2}$$
$$H(z) = \left(-\frac{(z-1)^2}{z}\right)\left(\frac{z^2}{(z-1)^2}\right)$$
$$H(z) = -z \quad \text{ROC: } 0 \le |z| < \infty$$

ROC includes the unit circle, hence overall $h[n]$ system is BIBO stable. even though $h_2[n]$ is unstable, the overall system cancels the pole/zero factors, and leads a stable overall system.

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Solution:

We are considering the following LCCDE

$$y[n] = x[n] + \frac{5}{2}x[n-1] + x[n-2]$$

(a) The impulse response can be derived by plugging in $\delta[n]$ for $x[n]$

$$h[n] = \delta[n] + \frac{5}{2}\delta[n-1] + \delta[n-2]$$

and the transfer function from the Z-transform of $h[n]$

$$H(z) = 1 + \frac{5}{2}z^{-1} + z^{-2}$$

The ROC is $z \neq 0$.

(b) The inverse system to $H(z)$ is given by the following

$$G(z) = \frac{1}{H(z)} = \frac{1}{1 + \frac{5}{2}z^{-1} + z^{-2}}$$

Because we want the *causal* inverse system, we keep everything in terms of *negative* powers of z. The difference equation can then be derived as

$$G(z) = \frac{S(z)}{Y(z)} = \frac{1}{1 + \frac{5}{2}z^{-1} + z^{-2}}$$
$$S(z)(1 + \frac{5}{2}z^{-1} + z^{-2}) = Y(z)$$
$$S(z) + \frac{5}{2}z^{-1}S(z) + z^{-2}S(z) = Y(z)$$
$$s[n] + \frac{5}{2}s[n-1] + s[n-2] = y[n]$$
$$s[n] = -\frac{5}{2}s[n-1] - s[n-2] + y[n]$$

(c) From part (a), we saw the ROC of $H(z)$ was $z \neq 0$, which contains the unit circle. Therefore, $H(z)$ is **stable**. For $G(z)$, we need to find the poles.

$$G(z) = \frac{1}{1 + \frac{5}{2}z^{-1} + z^{-2}} = \frac{1}{(1 + \frac{1}{2}z^{-1})(1 + 2z^{-1})}$$

$G(z)$ has poles at $z = -\frac{1}{2}, -2$. Because we chose $G(z)$ to be causal, its ROC is $|z| > 2$. Therefore, $G(z)$ is **not stable**.

(d) After factoring $G(z)$ in part (c), we can easily split it into two terms where $G_1(z) = \frac{1}{1 + \frac{1}{2}z^{-1}}$ and $G_2(z) = \frac{1}{1 + 2z^{-1}}$. The ROC of $G_1(z)$ is $|z| > \frac{1}{2}$, so it is **stable** and casual. For $G_2(z)$ to be stable, we need to implement it as an anti-causal system, so its ROC becomes $|z| < 2$.

For the difference equations, let $Y(z)$ represent the input to system $G_1(z)$, let $W(z)$ represent the output

of system $G_1(z)$ which is also the input to $G_2(z)$, and let $S(z)$ represent the output of system $G_2(z)$.

$$G_1(z) = \frac{W(z)}{Y(z)} = \frac{1}{1 + \frac{1}{2}z^{-1}}$$

$$W(z)(1 + \frac{1}{2}z^{-1}) = Y(z)$$

$$W(z) + \frac{1}{2}z^{-1}W(z) = Y(z)$$

$$w[n] + \frac{1}{2}w[n-1] = y[n]$$

$$w[n] = -\frac{1}{2}w[n-1] + y[n]$$

$$G_2(z) = \frac{S(z)}{W(z)} = \frac{1}{1 + 2z^{-1}} = \frac{z}{z+2}$$

$$S(z)(z+2) = zW(z)$$

$$zS(z) + 2S(z) = zW(z)$$

$$s[n+1] + 2s[n] = w[n+1]$$

$$s[n] = \frac{w[n+1] - s[n+1]}{2}$$

Grading: *20 points*

- $+5$ pts for each correct part