HW Solutions

ECE 310: Digital Signal Processing, Spring 2026 Version: **1.0**

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Solution:

(a) $x_1[n] = \delta[n+3] + 4\delta[n] - \delta[n-2]$.

We can use the following z-transform pair

$$\delta[n] \xleftrightarrow{\mathcal{Z}} 1$$

with the time-shifting property of the z-transform to get

$$X_1(z) = z^3 + 4 - z^{-2}.$$

The above expression diverges for $|z| = 0$ and $|z| = \infty$ due to the positive and negative delays present, so our ROC is

$$\text{ROC:} \quad 0 < |z| < \infty.$$

(b) $x_2[n] = \left(\frac{3}{4}\right)^{n+3} u[n-2]$.

We start with,

$$x_2[n] = \left(\frac{3}{4}\right)^3 \left(\frac{3}{4}\right)^n u[n-2]$$

Therefore,

$$X_2(z) = \left(\frac{3}{4}\right)^3 \sum_{n=-\infty}^{\infty} \left(\frac{3}{4}\right)^n z^{-n} u[n-2] = \left(\frac{3}{4}\right)^3 \sum_{n=2}^{\infty} \left(\frac{3}{4} z^{-1}\right)^n.$$

Let $k = n - 2$, so that

$$X_2(z) = \left(\frac{3}{4}\right)^5 z^{-2} \sum_{k=0}^{\infty} \left(\frac{3}{4} z^{-1}\right)^k.$$

Here we recognize a geometric series. Recall that

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \quad \text{valid when } |a| < 1.$$

Applying this with $a = (3/4)z^{-1}$, we obtain

$$X_2(z) = \frac{(3/4)^3 z^{-2}}{1 - (3/4)z^{-1}}.$$

Finally, simplifying the numerator gives

$$X_2(z) = \frac{(3/4)^5 \, z^{-2}}{1 - (3/4)z^{-1}}.$$

From the geometric series convergence condition, we require

$$\left|\tfrac{3}{4}z^{-1}\right| < 1 \quad \Rightarrow \quad |z| > \tfrac{3}{4}.$$

Alternative approach: Alternatively, we can use common z-transform pairs to solve this problem. We can notice that $x_2[n]$ resembles a signal of the form $a^n u[n]$, however the delay in the exponent does not match the delay in the unit step. Without shifting or scaling, the base function has the known z-transform:

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}} \qquad \text{ROC:} \quad |z| > |a|$$

To get the signal to match the desired form, we can do the following:

$$\left(\frac{3}{4}\right)^{n+3} u[n-2] = \left(\frac{3}{4}\right)^5 \left(\frac{3}{4}\right)^{n-2} u[n-2]$$

Let $G(z)$ be the z-transform of $\left(\frac{3}{4}\right)^n u[n]$:

$$G(z) = \frac{1}{1-\frac{3}{4}z^{-1}} \qquad \text{ROC:} \quad |z| > \frac{3}{4}$$

By the linearity and time-shifting properties of the z-transform, we have

$$X_2(z) = \left(\frac{3}{4}\right)^5 z^{-2}G(z) = \left(\frac{3}{4}\right)^5 \frac{z^{-2}}{1-\frac{3}{4}z^{-1}}.$$

The ROC of the expression above does not include $|z| = 0$, which happens to already be implied by the ROC of $G(z)$:

$$\text{ROC:} \quad |z| > \frac{3}{4}$$

(c) $x_3[n] = 3^n u[-n] + 2^{-n} u[n]$**.**

Split into left/right sided parts and use the geometric series properties again, we have:

$$X_{3a}(z) = \sum_{n=-\infty}^{0} 3^n z^{-n} = \sum_{k=0}^{\infty} 3^{-k} z^k = \sum_{k=0}^{\infty} (\frac{z}{3})^k = \frac{1}{1-\frac{z}{3}}, \quad |z| < 3.$$

$$X_{3b}(z) = \sum_{n=0}^{\infty} (2^{-1}z^{-1})^n = \frac{1}{1-(1/2)z^{-1}}, \quad |z| > \tfrac{1}{2}.$$

Thus

$$X_3(z) = \frac{1}{1-(1/3)z} + \frac{1}{1-(1/2)z^{-1}},$$

These ROCs overlap, leaving us with an overall ROC of $\frac{1}{2} < |z| < 3$.

Alternative approach: Once again, we can use transform pairs and properties instead. Let's alter these two terms so that they have a form that we can apply properties to:

$$3^n u[-n] + 2^{-n} u[n] = \left(\frac{1}{3}\right)^{-n} u[-n] + \left(\frac{1}{2}\right)^n u[n]$$

While the second term straightforwardly matches a transform pair, for the first term we need to use the time-reversal (or folding) property. Let $A(z)$ be the z-transform of $\left(\frac{1}{3}\right)^n u[n]$:

$$A(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} \qquad \text{ROC:} \quad |z| > \frac{1}{3}$$

To apply the time-reversal property, we need to find $A(z^{-1})$ and replace z with z^{-1} in the ROC. This gives us

$$A(z^{-1}) = \frac{1}{1 - \frac{1}{3}z} \qquad \text{ROC:} \quad |z^{-1}| > \frac{1}{3} \quad \to |z| < 3.$$

Thus:

$$X_3(z) = \frac{1}{1 - \frac{1}{3}z} + \frac{1}{1 - \frac{1}{2}z^{-1}}$$

The ROC of $X_3(z)$ is the intersection of the two terms' ROCs:

$$\text{ROC:} \quad \frac{1}{2} < |z| < 3$$

(d) $x_4[n] = \left(\frac{1}{4}\right)^{|n|}$**.**

Break the signal at $n = 0$ and use the the geometric series properties:

$$n \geq 0, X_{4a}(z) = \sum_{n=0}^{\infty} \left(\tfrac{1}{4}z^{-1}\right)^n = \frac{1}{1 - (1/4)z^{-1}}, \quad |z| > \tfrac{1}{4};$$

$$n \leq -1, X_{4b}(z) = \sum_{n=-\infty}^{-1} \left(\tfrac{1}{4}\right)^{-n} z^{-n} = \sum_{k=1}^{\infty} \left(\tfrac{1}{4}\right)^k z^k = \frac{(1/4)z}{1 - (1/4)z}, \quad |z| < 4.$$

So

$$X_4(z) = X_{4a}(z) + X_{4b}(z) = \frac{1}{1 - (1/4)z^{-1}} + \frac{(1/4)z}{1 - (1/4)z}, \qquad \text{ROC: } \tfrac{1}{4} < |z| < 4.$$

Alternative approach: We can also solve this question using known z-transforms. Since we have two different functions over two different domains, we can rewrite this signal using step functions:

$$x_4[n] = \left(\frac{1}{4}\right)^n u[n] + \left(\frac{1}{4}\right)^{-n} u[-n-1]$$

$$= \left(\frac{1}{4}\right)^n u[n] + 4^n u[-n-1]$$

The first term is of the familiar form $a^n u[n]$, while the second term matches the form of its left-sided counterpart, $-a^n u[-n-1]$:

$$-a^n u[-n-1] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}} \qquad \text{ROC:} \quad |z| < |a|$$

Applying these transform pairs gives us

$$X_4(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} - \frac{1}{1 - 4z^{-1}}.$$

Once again, the ROC of $X_4(z)$ is the intersection of the ROCs of its two terms:

$$\text{ROC:} \quad \frac{1}{4} < |z| < 4$$

It should be noted...

$$\frac{\alpha z}{1 - \alpha z} = \frac{\alpha}{z^{-1} - \alpha} = \frac{1}{\frac{1}{\alpha}z^{-1} - 1} = -\frac{1}{1 - \frac{1}{\alpha}z^{-1}}$$

so this answer is equivalent to the answer obtained with the geometric series approach.

(e) $x_5[n] = n\left(\frac{1}{2}\right)^n \big(u[n] - u[n-5]\big)$**.**

Since this is a finite-length signal, we can rewrite it in terms of shifted and scaled copies of $\delta[n]$:

$$n\left(\frac{1}{2}\right)^n (u[n] - u[n-5]) = n\left(\frac{1}{2}\right)^n (\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4])$$

$$= \frac{1}{2}\delta[n-1] + \frac{1}{2}\delta[n-2] + \frac{3}{8}\delta[n-3] + \frac{1}{4}\delta[n-4]$$

The last step above uses the sampling property of the delta function (specifically, $g[n]\delta[n-n_0] = g[n_0]\delta[n-n_0]$). Now, we can easily take the z-transform of this expression like we did in part (a):

$$X_5(z) = \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2} + \frac{3}{8}z^{-3} + \frac{1}{4}z^{-4}$$

Only for $|z| = 0$ does this expression diverge, so our ROC is

$$\text{ROC:} \quad |z| > 0.$$

Grading: *30 points, 6 points each*

- For each of (a), (b), (c), (d), and (e):
  - +6: Correctly applied z-transform formula or transform pairs to find correct answer (-1 point for each minor math mistake)
  - +3: Correct z-transform but ROC is incorrect / missing
  - +0: Empty or invalid

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Solution:

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} \qquad \text{ROC:} \quad |z| > \frac{1}{3}$$

(a) $x_1[n] = x[n+3]$**.**

Using the time-shifting property, we get

$$X_1(z) = z^3 X(z) = \frac{z^3}{1 - \frac{1}{3}z^{-1}}$$

Since we now have z^3 in the numerator, our ROC can no longer include $|z| = \infty$:

$$\text{ROC:} \quad \frac{1}{3} < |z| < \infty$$

(b) $x_2[n] = 2^n x[n]$**.**

Here, we can use the scaling property:

$$a^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(a^{-1}z) \qquad \text{ROC:} \quad |a|R_x$$

$$\begin{aligned} \rightarrow X_2(z) &= X(2^{-1}z) \\ &= \frac{1}{1 - \frac{1}{3}\left(\frac{z}{2}\right)^{-1}} \\ &= \frac{1}{1 - \frac{2}{3}z^{-1}} \end{aligned}$$

This gives us an ROC of

$$\text{ROC:} \quad |z| > \frac{2}{3}.$$

(c) $x_3[n] = \cos(\frac{\pi}{4}n)\, x[n]$**.**

We can replace the cosine with exponentials using the Euler identity:

$$\cos(x) = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$$

This gives us

$$\begin{aligned} \cos\left(\frac{\pi}{4}n\right) x[n] &= \frac{1}{2}\left(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}\right) x[n] \\ &= \frac{1}{2}\left(e^{j\frac{\pi}{4}n}x[n] + e^{-j\frac{\pi}{4}n}x[n]\right). \end{aligned}$$

Now, we can utilize the same scaling property as before with $a = e^{j\frac{\pi}{4}}$ for the first term and $a = e^{-j\frac{\pi}{4}}$ for the second term:

$$\begin{aligned} X_3(z) &= \frac{1}{2}\left(X\left(e^{-j\frac{\pi}{4}}z\right) + X\left(e^{j\frac{\pi}{4}}z\right)\right) \\ &= \frac{1}{2}\left(\frac{1}{1 - \frac{1}{3}e^{j\frac{\pi}{4}}z^{-1}} + \frac{1}{1 - \frac{1}{3}e^{-j\frac{\pi}{4}}z^{-1}}\right) \end{aligned}$$

Because $\left|e^{j\frac{\pi}{4}}\right| = \left|e^{-j\frac{\pi}{4}}\right| = 1$, our ROC remains unchanged:

$$\text{ROC:} \quad |z| > \frac{1}{3}$$

(d) $x_4[n] = n(n-2)\, x[n-1]$**.**

First, we can simplify the expression by distributing:

$$n(n-2)x[n-1] = n^2 x[n-1] - 2nx[n-1]$$

Let $G(z)$ be the z-transform of $x[n-1]$:

$$\begin{aligned} G(z) &= z^{-1}X(z) \\ &= \frac{z^{-1}}{1-\frac{1}{3}z^{-1}} \\ &= \frac{1}{z-\frac{1}{3}} \end{aligned}$$

We take that last simplification step to make future calculations for this problem easier. The ROC is therefore:

$$\text{ROC:} \quad |z| > \frac{1}{3}$$

Using the differentiation property

$$nx[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz} \qquad \text{ROC:} \quad R_x$$

and product rule, we can find the following z-transforms,

$$\begin{aligned} nx[n-1] &\overset{\mathcal{Z}}{\longleftrightarrow} -zG'(z) \\ n^2x[n-1] &\overset{\mathcal{Z}}{\longleftrightarrow} zG'(z) + z^2G''(z) \end{aligned}$$

where $G'(z)$ and $G''(z)$ denote the first and second derivatives of $G(z)$ with respect to z. We can now express $X_4(z)$ as

$$\begin{aligned} X_4(z) &= \left(zG'(z) + z^2G''(z)\right) - 2\left(-zG'(z)\right) \\ &= 3zG'(z) + z^2G''(z). \end{aligned}$$

Solving for $G'(z)$ and $G''(z)$ gives us:

$$G'(z) = -\frac{1}{\left(z-\frac{1}{3}\right)^2}$$

$$G''(z) = \frac{2}{\left(z-\frac{1}{3}\right)^3}$$

Plugging into our expression for $X_4(z)$ gives us

$$\begin{aligned} X_4(z) &= -\frac{3z}{\left(z-\frac{1}{3}\right)^2} + \frac{2z^2}{\left(z-\frac{1}{3}\right)^3} \\ &= \frac{z-z^2}{\left(z-\frac{1}{3}\right)^3}. \end{aligned}$$

Since this is a proper rational function, it will not diverge if $|z| = \infty$, so our ROC remains unchanged:

$$\text{ROC:} \quad |z| > \frac{1}{3}$$

Alternatively, we could have started with

$$G(z) = \frac{3}{3z-1},$$

which leads to an equivalent final answer of

$$\begin{aligned} X_4(z) &= -\frac{27z}{(3z-1)^2} + \frac{54z^2}{(3z-1)^3} \\ &= \frac{27z - 27z^2}{(3z-1)^3}. \end{aligned}$$

(e) $x_5[n] = \left(\frac{1}{5}\right)^n u[n] \; * \; x[n]$**.**

The presence of the convolution operator indicates that we should use the convolution property of the z-transform:

$$x_1[n] * x_2[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X_1(z)X_2(z) \qquad \text{ROC:} \quad \text{At least} \quad R_{x_1} \cap R_{x_2}$$

Let $G(z)$ be the z-transform of $\left(\frac{1}{5}\right)^n u[n]$:

$$G(z) = \frac{1}{1 - \frac{1}{5}z^{-1}} \qquad \text{ROC:} \quad |z| > \frac{1}{5}$$

By the convolution property, we have

$$\begin{aligned} X_5(z) &= G(z)X(z) \\ &= \frac{1}{\left(1 - \frac{1}{5}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)} \end{aligned}$$

with an ROC of

$$\text{ROC:} \quad |z| > \frac{1}{3}.$$

Grading: *30 points, 6 points each*

- For each of (a), (b), (c), (d), and (e):
  - -0.0: Correct.
  - -1.5: Z-transform: Minor numerical mistake
  - -3.0: Z-transform: Major mistake
  - -4.0: Z-transform: wrong without a valid attempt
  - -1.0: ROC: minor mistake (missing one side of the boundary, lack of the abs bracket $|\cdot|$, etc)
  - -1.0: ROC: major mistake without a valid attempt
  - -6.0: Empty or invalid

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Solution:

(a) $H_1(z) = 1 - \frac{2}{3}z^{-1} + \frac{4}{9}z^{-2} - \frac{8}{27}z^{-3}$**.**

Once we recognize each of these terms corresponds to a shifted and scaled $\delta[n]$, we can use the linearity and time-shifting properties of the z-transform to get

$$\begin{aligned} h_1[n] &= \delta[n] - \frac{2}{3}\delta[n-1] + \frac{4}{9}\delta[n-2] - \frac{8}{27}\delta[n-3] \\ &= \left(-\frac{2}{3}\right)^n (u[n] - u[n-4]). \end{aligned}$$

(b) $H_2(z) = \dfrac{3z^{-2}}{1+2z^{-1}}$, **ROC:** $|z| > 2$**.**

Let $A(z) = \frac{1}{1+2z^{-1}}$ with an ROC of $|z| > 2$. From our transform pairs, we see then that the inverse z-transform of $A(z)$ is $a[n] = (-2)^n u[n]$, which is a right-sided signal.

According to the linearity and time-shifting properties:

$$\begin{aligned} h_2[n] &= 3a[n-2] \\ &= 3(-2)^{n-2}u[n-2] \end{aligned}$$

(c) $H_3(z) = \dfrac{1}{1-(1/4)z^{-1}} + \dfrac{1}{1-(3/2)z^{-1}}$, **ROC:** $\frac{1}{4} < |z| < \frac{3}{2}$**.**

First, we should use the ROC to determine whether each term corresponds to a left-sided or right-sided signal.

We can see that the ROC exists inside $|z| = \frac{3}{2}$, indicating that the second term is left-sided. Similarly, we can see that the ROC exists outside $|z| = \frac{1}{4}$, indicating that the first term is right-sided.

Using the corresponding transform pairs, we get

$$h_3[n] = \left(\frac{1}{4}\right)^n u[n] - \left(\frac{3}{2}\right)^n u[-n-1].$$

(d) $H_4(z) = \dfrac{1}{\left(1-(\frac{2}{3})z^{-1}\right)(1-z^{-1})}$, **ROC:** $|z| > 1$**.**

Because we have two polynomial factors multiplied together in the denominator, we should use partial fraction decomposition to separate the expression into multiple terms:

$$\begin{aligned} \frac{1}{\left(1-\frac{2}{3}z^{-1}\right)(1-z^{-1})} &= \frac{A_1}{1-\frac{2}{3}z^{-1}} + \frac{A_2}{1-z^{-1}} \\ &\rightarrow 1 = A_1(1-z^{-1}) + A_2\left(1-\frac{2}{3}z^{-1}\right) \end{aligned}$$

Setting $z^{-1} = \frac{3}{2}$ gives us:

$$\begin{aligned} 1 &= A_1\left(1-\frac{3}{2}\right) \\ &\rightarrow A_1 = -2 \end{aligned}$$

Setting $z^{-1} = 1$ gives us:

$$1 = A_2 \left(1 - \frac{2}{3}\right)$$
$$\rightarrow A_2 = 3$$

Our decomposed $H_4(z)$ is then

$$H_4(z) = -\frac{2}{1 - \frac{2}{3}z^{-1}} + \frac{3}{1 - z^{-1}}.$$

Based on the ROC given, we see that both terms must be right-sided. Thus, the inverse transform is

$$h_4[n] = -2\left(\frac{2}{3}\right)^n u[n] + 3u[n]$$
$$= \left(-2\left(\frac{2}{3}\right)^n + 3\right) u[n].$$

Grading: *20 points, 6 points each*

- For each of (a), (b), (c), and (d):
  - -0.0: Correct.
  - -2.0: Minor mistake: minor numerical mistake in 1-2 terms
  - -4.0: Major mistake: multiple numerical mistakes or invalid attempt with effort shown
  - -6.0: Empty or invalid

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Solution:

(a)

To find the zeros of $H(z)$, we simply set the numerator equal to zero and solve for z:

$$1 - 3z^{-1} = 0$$
$$\rightarrow z = 3$$

Similarly, we set the denominator equal to zero to solve for the poles:

$$1 + \frac{1}{6}z^{-1} - \frac{1}{3}z^{-2} = 0$$

To factor the left side, we can set $x = z^{-1}$:

$$-\frac{1}{3}x^2 + \frac{1}{6}x + 1 = 0$$
$$\rightarrow -2x^2 + x + 6 = 0$$
$$\rightarrow (-2x - 3)(x - 2) = 0$$

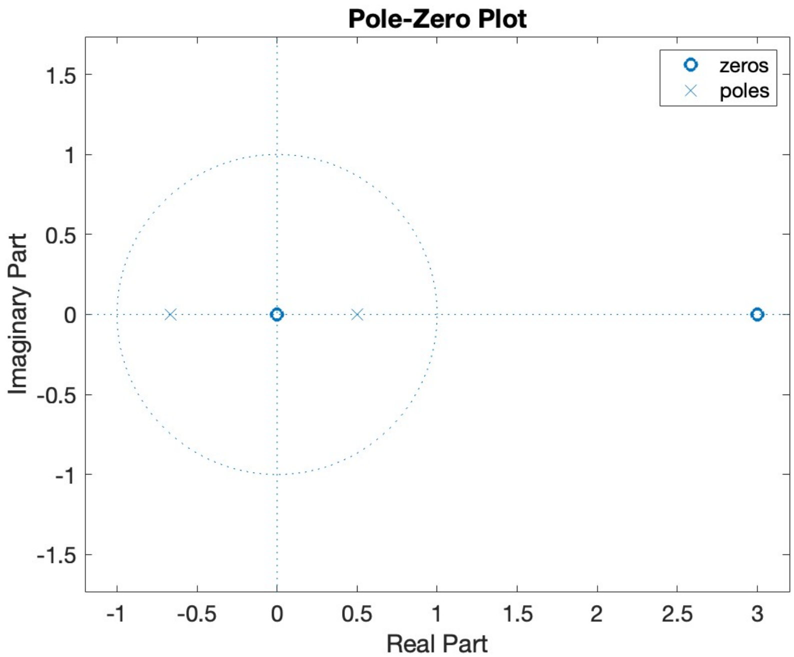
Thus, $x = 2, -\frac{3}{2}$, or equivalently, $z = \frac{1}{2}, -\frac{2}{3}$.

(b)

Right–sided system $\Rightarrow$ ROC is outside the outermost pole:

$$\text{ROC: } |z| > \max\{|-\tfrac{2}{3}|, |\tfrac{1}{2}|\} = \tfrac{2}{3}.$$

The pole-zero plot is shown as below.



(c)

To factor the denominator, we can do the following:

$$\begin{aligned} H(z) &= \frac{1-3z^{-1}}{\frac{1}{6}(6+z^{-1}-2z^{-2})} \\ &= \frac{1-3z^{-1}}{\frac{1}{6}(-2z^{-1}-3)(z^{-1}-2)} \\ &= \frac{1-3z^{-1}}{\left(1+\frac{2}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \end{aligned}$$

Performing partial fraction decomposition gives us

$$\begin{aligned} H(z) &= \frac{A_1}{1+\frac{2}{3}z^{-1}} + \frac{A_2}{1-\frac{1}{2}z^{-1}} \\ \rightarrow A_1 &= \frac{22}{7} \\ \rightarrow A_2 &= -\frac{15}{7} \\ \rightarrow h[n] &= \frac{22}{7}\left(-\frac{2}{3}\right)^n u[n] - \frac{15}{7}\left(\frac{1}{2}\right)^n u[n]. \end{aligned}$$

Note that we chose both terms to be right-sided because the problem tells us that the impulse response is right-sided.

Grading: *16 points*

- For part (a) (5 points):
  - -0: Correct
  - -1: Minor mistake
  - -3: Major mistake or partially wrong in both zeros and poles
  - -5: Empty or invalid
- For part (b) (5 points):
  - -0: Correct
  - -1: ROC: Minor mistake (missing one side of the boundary, lack of the abs bracket $|\cdot|$, etc)
  - -2: ROC: Major mistake without a valid attempt
  - -1: Plot: Wrong zeros
  - -1: Plot: Wrong poles
  - -1: Plot: Wrong ROC
  - -5: Empty or invalid
- For part (c) (6 points):
  - -0.0: Correct.
  - -2.0: Minor mistake: minor numerical mistake in 1-2 terms
  - -4.0: Major mistake: multiple numerical mistakes or invalid attempt with effort shown
  - -6.0: Empty or invalid