# HW2 Solutions

**ECE 310: Digital Signal Processing, Spring 2026** Version: **1.0**

Due Date: 02/06/2026
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## 1

### Solution:

**a)** $y[n] = x[n+1] - 2x[n-2]$

- **Linearity:** Let $T$ be the system. We test for superposition:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= (ax_1[n+1] + bx_2[n+1]) - 2(ax_1[n-2] + bx_2[n-2]) \\ &= a(x_1[n+1] - 2x_1[n-2]) + b(x_2[n+1] - 2x_2[n-2]) \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

  The system is **linear**.
- **Time invariance:** Let $y'[n] = T(x[n-n_0]) = x[n-n_0+1] - 2x[n-n_0-2]$.
  The shifted output is $y[n-n_0] = x[(n-n_0)+1] - 2x[(n-n_0)-2]$.
  Since $y'[n] = y[n-n_0]$, the system is **time-invariant**.
- **Causality:** The output $y[n]$ depends on $x[n+1]$, which is a future value of the input. Thus, the system is **non-causal**.

**b)** $y[n] = 5x[-n] + 1$

- **Linearity:** A system with a non-zero constant offset (+1) is non-linear. Specifically, $T(0) = 1 \neq 0$. Therefore, the system is **non-linear**.
- **Time invariance:** Let $y'[n] = T(x[n-n_0]) = 5x[-n-n_0] + 1$.
  The shifted output is $y[n-n_0] = 5x[-(n-n_0)] + 1 = 5x[-n+n_0] + 1$.
  Since $y'[n] \neq y[n-n_0]$, the system is **time-varying**.
- **Causality:** For $n = -1$, $y[-1] = 5x[1] + 1$. The output depends on a future value of the input. Thus, the system is **non-causal**.

**c)** $y[n] = x[3] \cdot x[n]$

- **Linearity:**

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= (ax_1[3] + bx_2[3]) \cdot (ax_1[n] + bx_2[n]) \\ &= a^2 x_1[3]x_1[n] + abx_1[3]x_2[n] + abx_2[3]x_1[n] + b^2 x_2[3]x_2[n] \end{aligned}$$

  This does not equal $aT(x_1[n]) + bT(x_2[n])$. The system is **non-linear**.
- **Time invariance:** Let $y'[n] = T(x[n-n_0]) = x[3-n_0] \cdot x[n-n_0]$.
  The shifted output is $y[n-n_0] = x[3] \cdot x[n-n_0]$.
  Since $y'[n] \neq y[n-n_0]$, the system is **time-varying**.
- **Causality:** For any $n < 3$, the output depends on $x[3]$, which is a future value. Thus, the system is **non-causal**.

**d)** $y[n] = e^{j\pi n}x[n]$

- **Linearity:** $T(ax_1[n] + bx_2[n]) = e^{j\pi n}(ax_1[n] + bx_2[n]) = a(e^{j\pi n}x_1[n]) + b(e^{j\pi n}x_2[n])$.
  This equals $aT(x_1[n]) + bT(x_2[n])$, so the system is $\boxed{\textbf{linear}}$.
- **Time invariance:** Let $y'[n] = T(x[n - n_0]) = e^{j\pi n}x[n - n_0]$.
  The shifted output is $y[n - n_0] = e^{j\pi(n-n_0)}x[n - n_0]$.
  Since the coefficient depends on $n$, $y'[n] \neq y[n - n_0]$, so the system is $\boxed{\textbf{time-varying}}$.
- **Causality:** The output $y[n]$ depends only on the current input $x[n]$. Thus, the system is $\boxed{\textbf{causal}}$.

**Grading:** *32 points, 8 points each*

- For each of (a), (b), (c), and (d):
  - +3: correctly identified linearity with proper justification
  - +3: correctly identified time-variance/invariance with proper justification
  - +2: correctly identified causality with proper justification

## 2

## Solution:

**a)** $x[n] = \{3, \underset{\uparrow}{4}, -2\}, h[n] = \{1, 0, \underset{\uparrow}{-2}, 0, -1\}$

First, we define the start indices and lengths:

$$n_s = -1, \quad n_e = 1, \quad N = 3$$
$$m_s = -2, \quad m_e = 2, \quad M = 5$$

The resulting signal $y[n] = x[n] * h[n]$ will have:

$$k_s = n_s + m_s = -3$$
$$k_e = n_e + m_e = 3$$
$$K = N + M - 1 = 7$$

We use matrix-vector multiplication where the matrix is formed by shifts of $h[n]$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & -2 & 0 \\ -1 & 0 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -8 \\ -8 \\ 1 \\ -4 \\ 2 \end{bmatrix}$$

$$\boxed{x[n] * h[n] = \{3, 4, -8, \underset{\uparrow}{-8}, 1, -4, 2\}}$$

**b)** $x[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1], h[n] = -n(u[n+2] - u[n-2])$

First, we simplify $h[n]$. The window $(u[n+2] - u[n-2])$ is non-zero only for $n \in \{-2, -1, 0, 1\}$.

$$\begin{aligned} h[n] &= -(-2)\delta[n+2] - (-1)\delta[n+1] - (0)\delta[n] - (1)\delta[n-1] \\ &= 2\delta[n+2] + \delta[n+1] - \delta[n-1] \end{aligned}$$

Using the property $x[n] * \delta[n-k] = x[n-k]$:

$$\begin{aligned} y[n] &= x[n] * (2\delta[n+2] + \delta[n+1] - \delta[n-1]) \\ &= 2x[n+2] + x[n+1] - x[n-1] \\ &= \boxed{2\left(\frac{1}{2}\right)^{n+1} u[n+1] + \left(\frac{1}{2}\right)^{n} u[n] - \left(\frac{1}{2}\right)^{n-2} u[n-2]} \end{aligned}$$

**c)** $x[n] = 2^n u[n], h[n] = \{1, \underset{\uparrow}{-1}, 2\}$

Rewriting $h[n]$ as a sum of impulses:

$$h[n] = \delta[n+1] - \delta[n] + 2\delta[n-1]$$

Applying the convolution property:

$$\begin{aligned} y[n] &= x[n+1] - x[n] + 2x[n-1] \\ &= 2^{n+1}u[n+1] - 2^n u[n] + 2(2^{n-1}u[n-1]) \\ &= \boxed{2^{n+1}u[n+1] - 2^n u[n] + 2^n u[n-1]} \end{aligned}$$

**d)** $x[n] = e^{j\pi n}u[n], h[n] = 2^{-n}u[n-2]$

Since both are infinite-length, we use the summation definition. Note $e^{j\pi n} = (-1)^n$.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} (-1)^k u[k] \cdot 2^{-(n-k)} u[n-k-2]$$

The summation limits are $k \geq 0$ and $k \leq n-2$. This is non-zero only for $n \geq 2$.

$$\begin{aligned} y[n] &= 2^{-n} \sum_{k=0}^{n-2} (-1)^k 2^k = 2^{-n} \sum_{k=0}^{n-2} (-2)^k \\ &= 2^{-n} \left[ \frac{1 - (-2)^{n-1}}{1 - (-2)} \right] \\ &= \boxed{\frac{2^{-n} - 2^{-n}(-2)^{n-1}}{3}} \end{aligned}$$

**Grading:** *24 points, 6 points each*

- For each of (a), (b), (c), and (d):
  - +6: correct answer (-1 for each minor math mistake)
  - +0: empty or invalid

## 3

## Solution:

Examining the plots of $x[n]$ and $y[n]$, we can express the input as:

$$x[n] = \delta[n-1] + 2\delta[n-4]$$

The output $y[n]$ contains two shifted and scaled copies for every input impulse. For the impulse at $n = 1$, the system produces samples at $n = 3$ (magnitude 1) and $n = 4$ (magnitude 0.5). This indicates a delay of 2 with gain 1, and a delay of 3 with gain 0.5.

We verify this with the second impulse at $n = 4$ (magnitude 2). The system produces $2 \cdot (1) = 2$ at $n = 4+2 = 6$ and $2 \cdot (0.5) = 1$ at $n = 4+3 = 7$, which matches the plot exactly. Therefore, the impulse response is:

$$\boxed{h[n] = \delta[n-2] + 0.5\delta[n-3]}$$

**Grading:** *15 points*

- +15: correct answer
- +12: minor mistake with clear reasoning/thought process
- +0: empty or invalid

## 4

## Solution:

As suggested in the hint, we begin by expressing the unit impulse $\delta[n]$ in terms of the unit step $u[n]$ as $\delta[n] = u[n] - u[n-1]$. Using this relation, we can rewrite the input $x[n] = \delta[n] - 5\delta[n-1]$ as:

$$\begin{aligned} x[n] &= (u[n] - u[n-1]) - 5(u[n-1] - u[n-2]) \\ &= u[n] - 6u[n-1] + 5u[n-2] \end{aligned}$$

Since the system is LTI, the output $y[n]$ is the same linear combination of the step response $g[n]$ and its shifted versions. Given $g[n] = \left(\frac{1}{3}\right)^n u[n]$, we have:

$$\begin{aligned} y[n] &= g[n] - 6g[n-1] + 5g[n-2] \\ &= \left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{3}\right)^{n-1} u[n-1] + 5\left(\frac{1}{3}\right)^{n-2} u[n-2] \quad (*) \end{aligned}$$

We can simplify and factor out $\left(\frac{1}{3}\right)^n$ from each term (this step is not necessary but shows an alternative form):

$$\begin{aligned} y[n] &= \left(\frac{1}{3}\right)^n u[n] - 18\left(\frac{1}{3}\right)^n u[n-1] + 45\left(\frac{1}{3}\right)^n u[n-2] \quad (*) \\ &= \boxed{\left(\frac{1}{3}\right)^n (u[n] - 18u[n-1] + 45u[n-2])}. \quad (*) \end{aligned}$$

All (*) are acceptable solutions.

**Grading:** *19 points*

- +19: correct answer
- +16: minor math mistake with correct LTI reasoning
- +8: correctly related $\delta[n]$ and $u[n]$ but failed to apply superposition properly
- +0: empty or invalid

## 5

## Solution:

To find the impulse response $h[n]$, we set $x[n] = \delta[n]$. Given zero initial conditions ($y[m] = 0$ for $m < 0$), the system equation becomes:

$$h[n] = \frac{1}{4}e^{j\frac{\pi}{3}}h[n-1] + \delta[n]$$

Evaluating for the first few values of $n$:

- For $n = 0$: $h[0] = \frac{1}{4}e^{j\frac{\pi}{3}}(0) + 1 = 1$
- For $n = 1$: $h[1] = \frac{1}{4}e^{j\frac{\pi}{3}}(1) + 0 = \frac{1}{4}e^{j\frac{\pi}{3}}$
- For $n = 2$: $h[2] = \frac{1}{4}e^{j\frac{\pi}{3}}\left(\frac{1}{4}e^{j\frac{\pi}{3}}\right) = \left(\frac{1}{4}e^{j\frac{\pi}{3}}\right)^2$

By induction, for $n \geq 0$, the pattern follows a geometric series. We can express the general form using the unit step function $u[n]$:

$$\boxed{h[n] = \left(\frac{1}{4}e^{j\frac{\pi}{3}}\right)^n u[n]}$$

**Grading:** *10 points*

- +10: Correct impulse response for (a)
- -2: Minor math or notation error
- +0: Empty or invalid