HW8 Solutions

ECE 310: Digital Signal Processing, Spring 2026 Version: **1.0**

Due Date: 5/6/26

1

Solution:

(a) From the graph, we see that the filter has length $N = 11$ and also has even symmetry, therefore this must be a **GLP Type-I Filter**

(b) We know that an ideal lowpass filter is constructed using $\frac{sin(\omega_c n)}{\pi n}$, with a peak value of $\frac{\omega_c}{\pi}$ where ω_c is the cutoff frequency.
Since the maximum value of this filter is $\frac{1}{3}$, $\omega_c = \frac{\pi}{3}$
Let's recall the methods to construct other ideal filters (which would then be windowed to create a realizable filter):

$$H_{highpass}(e^{j\omega}) = 1 - H_{lowpass}(e^{j\omega})$$

$$h_{bandpass}[n] = h_{lowpass}[n] \cdot cos(\omega_0 n)$$

$$H_{bandstop}(e^{j\omega}) = 1 - H_{bandpass}(e^{j\omega})$$

We can see this filter is only a windowed, time-shifted sinc function (time-shifted to ensure the causality of the filter), meaning the resulting filter will be a **Lowpass Filter**

Grading: *10 points*

- +5 each for each correct answer

2

Solution:

(a) From the graph, we see that there are multiple local π jumps, which means this filter cannot be strictly linear phase.

(b) The slope is the phase plot is

$$-\alpha = -\frac{N-1}{2}$$

From $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ the plot decreases by $-\frac{7\pi}{2}$, thus we can write

(c) There are no discontinuities at the origin of this graph, meaning the impulse response must have been constructed with cosines (giving us a choice between a Type-I and Type-II filter). Since our filter is even-length, this must be a **Type-II GLP Filter**. Note: Type-I should also be accepted if N=9 above.

(d) Since the phase makes a π jump at $\frac{\pi}{2}$, $A(\omega)$ must be changing from (+) to (-) (or vice-versa), this means that this function must be 0 at this point which means $|H_d(\frac{\pi}{2})| = 0$

Grading: *20 points*

- +5 for each correct answer
- full credit for part (b) if students have the correct relationship $-\alpha = -\frac{N-1}{2}$ but just read the slope from the graph differently.

3

Solution:

The impulse response $d[n]$ of an ideal low-pass filter with cutoff frequency ω_c is

$$d[n] = \frac{\sin(\omega_c n)}{\pi n}$$

which is of infinite length. Next, in order to obtain a causal, GLP filter with finite length N , we shift $d[n]$ by $\frac{N-1}{2}$ to the right and only focus on the value with $n = 0, 1, \ldots, N-1$, then we obtain

$$g[n] = d\left[n - \frac{N-1}{2}\right] = \frac{\sin\left(\omega_c\left(n - \frac{N-1}{2}\right)\right)}{\pi\left(n - \frac{N-1}{2}\right)}, \quad n = 0, 1, \ldots, N-1 \tag{1}$$

which has even symmetry. After that, we apply the window function $w[n]$ on $g[n]$ which gives

$$h[n] = w[n]g[n] = w[n] \cdot \frac{\sin\left(\omega_c\left(n - \frac{N-1}{2}\right)\right)}{\pi\left(n - \frac{N-1}{2}\right)}, \quad n = 0, 1, \ldots, N-1 \tag{2}$$

as our filter coefficients.

(a) Rectangular window: By definition,

$$w[n] = 1, \quad n = 0, 1, \ldots, N-1$$

then by plugging $w[n]$ and $\omega_c = \pi/3$ into Eq. (2) we get

$$h[n] = \frac{\sin\left(\frac{\pi}{3}\left(n - \frac{N-1}{2}\right)\right)}{\pi\left(n - \frac{N-1}{2}\right)}, \quad n = 0, 1, \ldots, N-1$$

(b) Hamming window: By definition,

$$w[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), \quad n = 0, 1, \ldots, N-1$$

then by plugging $w[n]$ and $\omega_c = \pi/3$ into Eq. (2) we get

$$h[n] = \left(0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)\right) \cdot \frac{\sin\left(\frac{\pi}{3}\left(n - \frac{N-1}{2}\right)\right)}{\pi\left(n - \frac{N-1}{2}\right)}, \quad n = 0, 1, \ldots, N-1$$

(c) The main difference between these two filters are on the transition band and the ripple. The filter results from the rectangular window has a thinner transition band but a larger ripple, in contrast, the filter from the Hamming window has a wider transition band but a smaller ripple.

Grading: *15 points*

- (a) 5 points
  - -1 if $(N-1)/2$ shift is missing
- (b) 5 points
  - -1 if $(N-1)/2$ shift is missing
  - -1 if the window expression is wrong
- (c) 5 points

4

Solution:

(a)

Yes. The impulse response $h[n]$ corresponds to a **Type-I linear phase filter**.

We first recognize that $h[n]$ has odd length, since $N = 25$. Furthermore, it consists of a Dirac delta and a sinc function, both centered at $n = 12$. Since $n = 12$ is the midpoint of the window and both functions are even-symmetric, $h[n]$ is even-symmetric. By definition, any filter with even symmetry and odd length is a Type-I linear phase filter.

(b)

To better analyze the filter's behavior, we will find $|H(\omega)|$, the magnitude of the DTFT of $h[n]$. We can see that $h[n] = g[n-12]$, where:

$$g[n] = \delta[n] - \frac{4}{5}\text{sinc}\left(\frac{4\pi}{5}n\right) \tag{3}$$

Using the appropriate transform pairs and the time shifting property of the DTFT, we get:

$$G(\omega) = 1 - \text{rect}\left(\frac{5}{8\pi}\omega\right) \tag{4}$$

$$\Rightarrow H(\omega) = G(\omega)e^{-j12\omega} \tag{5}$$

$$= \left(1 - \text{rect}\left(\frac{5}{8\pi}\omega\right)\right)e^{-j12\omega} \tag{6}$$

Thus, the magnitude of $H(\omega)$ is:

$$|H(\omega)| = 1 - \text{rect}\left(\frac{5}{8\pi}\omega\right) \tag{7}$$

When we check $|H(w)|$ and focus our attention to the range $\omega \in [0, \pi]$, we can see that the filter passes frequencies above $\frac{4\pi}{5}$ and removes frequencies below $\frac{4\pi}{5}$. Therefore, $h[n]$ is a **high-pass filter** with cutoff frequency $\omega_c = \frac{4\pi}{5}$.

(c)

Since we want to preserve the type and cutoff frequency of our original filter, $g[n]$ should still have even symmetry. This means that our Dirac delta and sinc function need to be centered at the midpoint of our new range, $0 \le n \le 74$, so:

$$g[n] = \delta[n-37] - \frac{4}{5}\text{sinc}\left(\frac{4\pi}{5}(n-37)\right), \quad 0 \le n \le 74 \tag{8}$$

Given our calculations in part (b) above, it should be easy to see that $|G(\omega)| = |H(\omega)|$. In other words, $g[n]$ is also a high-pass filter with cutoff frequency $\omega_c = \frac{4\pi}{5}$.

(d)

Just like in part (b), we want to find an expression for $|F(\omega)|$, the magnitude of the DTFT of $f[n]$. Once again, we can see that $f[n] = v[n-12]$, where:

$$v[n] = 2\left(\delta[n] - \frac{4}{5}\text{sinc}\left(\frac{4\pi}{5}n\right)\right)\cos\left(\frac{\pi}{2}n\right) \tag{9}$$

Using the modulation property of the DTFT, we get:

$$V(\omega) = 2\left(\frac{1}{2}\left(1 - \text{rect}\left(\frac{5}{8\pi}\left(\omega - \frac{\pi}{2}\right)\right)\right) + \frac{1}{2}\left(1 - \text{rect}\left(\frac{5}{8\pi}\left(\omega + \frac{\pi}{2}\right)\right)\right)\right) \tag{10}$$

$$= 2 - \text{rect}\left(\frac{5}{8\pi}\left(\omega - \frac{\pi}{2}\right)\right) - \text{rect}\left(\frac{5}{8\pi}\left(\omega + \frac{\pi}{2}\right)\right) \tag{11}$$

We know that time shifting will not affect the magnitude of $V(\omega)$, so $|F(\omega)| = |V(\omega)|$. After checking $|F(\omega)|$, we can see that $f[n]$ is a **band-pass filter** with low and high cutoff frequencies at $\omega_l = \frac{3\pi}{10}$ and $\omega_h = \frac{7\pi}{10}$, respectively.

(Note: Explicitly calculating the DTFT is not necessary for this part, as it can also be solved by first plotting $|H(\omega)|$ and graphically applying modulation to get $|F(\omega)|$.)

Grading: *20 points*

- +5 for each correct answer
- (a)

- $+3$: Declares GLP filter
- $+2$: Declares Type-I filter

- (b)
  - $+2$: Correctly identifies HPF
  - $+3$: Correctly identifies $\omega_c = \frac{4\pi}{5}$ cutoff
- (c)
  - $+5$: Correct expression for $g[n]$
- (d)
  - $+3$: Correctly identifies BPF
  - $+1$: Correctly identifies $\omega_l = \frac{3\pi}{10}$ lower cutoff
  - $+1$: Correctly identifies $\omega_h = \frac{7\pi}{10}$ higher cutoff

5

Solution:

(a)

We are downsampling by a factor of 3, which is represented in the time domain by

$$y[n] = x[3n]$$

. Thus,

$$y[n] = cos(\frac{\pi}{6} \cdot 3n) = cos(\frac{\pi}{2} n) = \{.., 1, 0, -1, 0, \underset{\uparrow}{1}, 0, -1, 0, 1, ...\}$$

(b)

We can approach this problem in the time domain by recognizing

$$X_d(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \leftrightarrow x[n] = (\frac{1}{2})^n u[n]$$

We can then apply the time domain definition of downsampling by D ($y[n] = x[Dn]$)...

$$y[n] = x[3n] = (\frac{1}{2})^{3n} u[3n] = (\frac{1}{8})^n u[n]$$

Then taking the IDTFT we find,

$$y[n] = (\frac{1}{8})^n u[n] \leftrightarrow Y_d(\omega) = \frac{1}{1 - \frac{1}{8}e^{-j\omega}}$$

Or we could approach this problem entirely in the frequency domain by using the frequency definition of downsampling. This work is as follows...

$$
\begin{aligned}
Y_d(\omega) &= \frac{1}{3}\sum_{k=0}^{2} X_d(\frac{\omega - 2\pi k}{3}) \\
&= \frac{1}{3}(X_d(\frac{\omega}{3}) + X_d(\frac{\omega}{3} - \frac{2\pi}{3}) + X_d(\frac{\omega}{3} - \frac{4\pi}{3})) \\
&= \frac{1}{3}(\frac{1}{1 - \frac{1}{2}e^{-j\frac{\omega}{3}}} + \frac{1}{1 - \frac{1}{2}e^{-j\frac{\omega}{3}}e^{j\frac{2\pi}{3}}} + \frac{1}{1 - \frac{1}{2}e^{-j\frac{\omega}{3}}e^{-j\frac{2\pi}{3}}}) \text{ (Note: } e^{-j\frac{2\pi}{3}} = e^{j\frac{4\pi}{3}}) \\
&= \frac{1}{3}(\frac{1}{1 - \frac{1}{2}e^{-j\frac{\omega}{3}}} + \frac{1 - \frac{1}{2}e^{-j\frac{\omega}{3}}e^{-j\frac{2\pi}{3}} + 1 - \frac{1}{2}e^{-j\frac{\omega}{3}}e^{j\frac{2\pi}{3}}}{(1 - \frac{1}{2}e^{-j\frac{\omega}{3}}e^{j\frac{2\pi}{3}})(1 - \frac{1}{2}e^{-j\frac{\omega}{3}}e^{-j\frac{2\pi}{3}})}) \\
&= \frac{1}{3}(\frac{1}{1 - \frac{1}{2}e^{-j\frac{\omega}{3}}} + \frac{2 - e^{-j\frac{\omega}{3}} \cdot \frac{1}{2}(e^{-j\frac{2\pi}{3}} + e^{j\frac{2\pi}{3}})}{1 - e^{-j\frac{\omega}{3}} \cdot \frac{1}{2}(e^{-j\frac{2\pi}{3}} + e^{j\frac{2\pi}{3}}) + \frac{1}{4}e^{-j\frac{2\omega}{3}}})
\end{aligned}
$$

Note: $\frac{1}{2}(e^{-j\frac{2\pi}{3}} + e^{j\frac{2\pi}{3}}) = cos(\frac{2\pi}{3}) = -\frac{1}{2}$

$$
\begin{aligned}
&= \frac{1}{3}(\frac{1 + \frac{1}{2}e^{-j\frac{\omega}{3}} + \frac{1}{4}e^{-j\frac{2\omega}{3}} + (2 + \frac{1}{2}e^{-j\frac{\omega}{3}})(1 - \frac{1}{2}e^{-j\frac{\omega}{3}})}{(1 - \frac{1}{2}e^{-j\frac{\omega}{3}})(1 + \frac{1}{2}e^{-j\frac{\omega}{3}} + \frac{1}{4}e^{-j\frac{2\omega}{3}})}) \\
&= \frac{1}{3}(\frac{1 + \frac{1}{2}e^{-j\frac{\omega}{3}} + \frac{1}{4}e^{-j\frac{2\omega}{3}} + 2 - e^{-j\frac{\omega}{3}} + \frac{1}{2}e^{-j\frac{\omega}{3}} - \frac{1}{4}e^{-j\frac{2\omega}{3}}}{1 + \frac{1}{2}e^{-j\frac{\omega}{3}} + \frac{1}{4}e^{-j\frac{2\omega}{3}} - \frac{1}{2}e^{-j\frac{\omega}{3}} - \frac{1}{4}e^{-j\frac{2\omega}{3}} - \frac{1}{8}e^{-j\omega}}) \\
&= \frac{1}{1 - \frac{1}{8}e^{-j\omega}}
\end{aligned}
$$

Grading: *20 points*

- (a) - 5 points
  - +5: Correct determination of downsampled signal
- (b) - 15 points
  - +5: Identify definition of downsampling (either frequency or time domain)
  - +10: Correct derivation of answer

6

Solution:

Check Figure 1

Grading: *15 points*

- (a), (b), & (c) - 5 points each
  - +1: Proper axes labels
  - +1: Correct amplitude scaling
  - +3: Correct spectrum shape

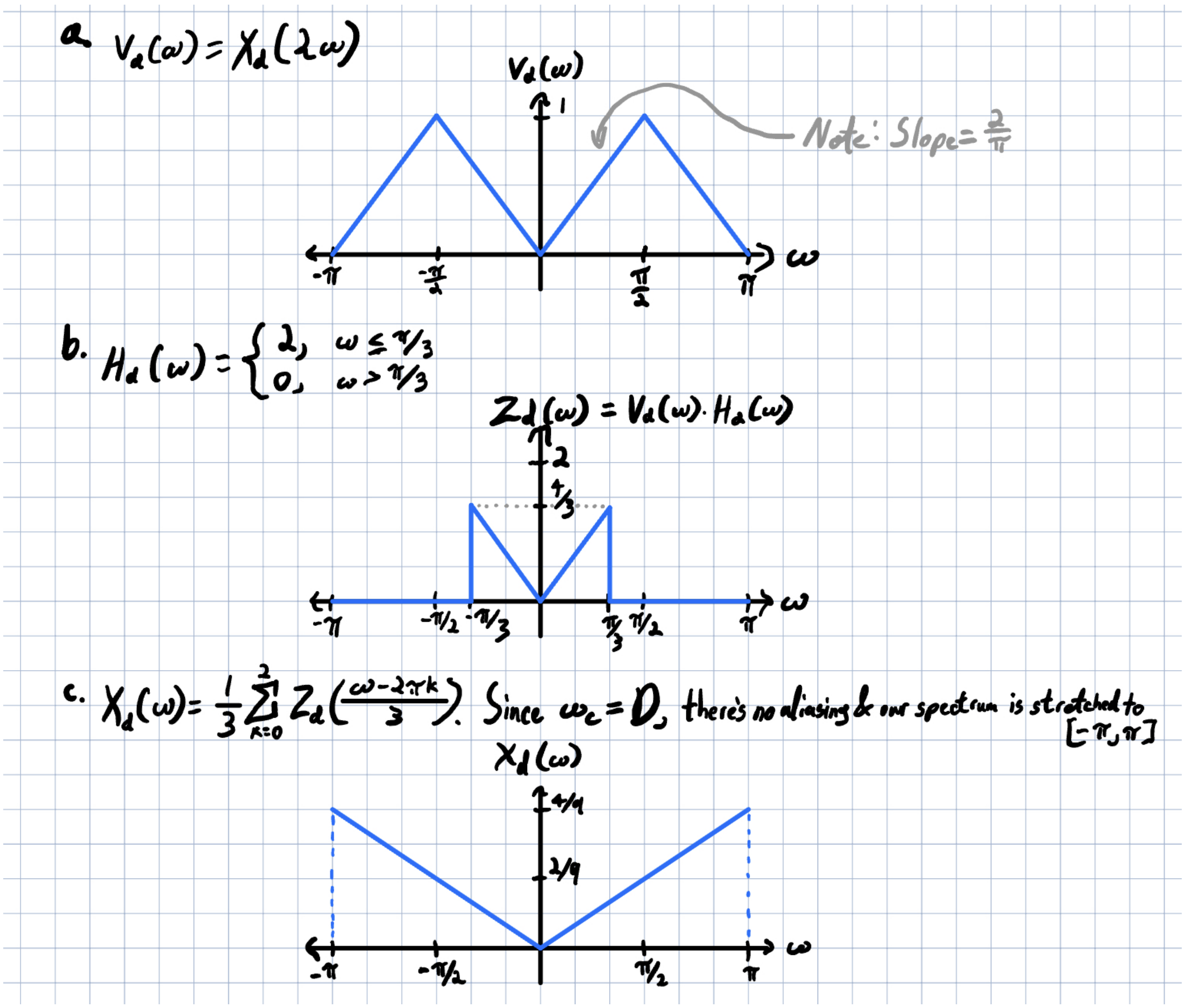

Figure 1: Diagrams for Question 6

7

Solution:

(a)

$p[n]$: 10,000 [samples/s] * 30 [s] = 300,000 samples

$v[n]$: 24,000 [samples/s] * 30 [s] = 720,000 samples

(b)

Check Figure 2

(c)

Check Figure 2

(d)

Since we converted both signals to have sampling rate of 30kHz, the total samples in the signal will be 900,000 samples. (We can also verify the conversion rates determined in (b)and (c) by noticing 3*300,000=900,000 and $\frac{5}{4}$*720,000=900,000

8

Solution:

(a)

i.

L=1, thus no upsampling occurs and $X_d(\omega) = Y_d(\omega)$. See Figure 3.

ii.

Since no upsampling occurred, we can use the original sampling period for T_Z. Thus, $T_Z = \frac{1}{200}$s

iii.

Theoretically the reconstruction filter transition bandwidth should be 0 for a ZOH DAC, although we would like to design the bandwidth to be as large as possible. We see that the edge of the DTFT of our signal is at $|\omega| = \frac{2\pi}{3}$, thus we can design our transition bandwidth to be the area between the edge of the non-central copy and the edge of the central copy.

$$\text{Transition Bandwidth} = \frac{\frac{4\pi}{3}}{T_Z} - \frac{\frac{2\pi}{3}}{T_Z} = \frac{400\pi}{3}$$

See Figure 4 for an illustration of this bandwidth.

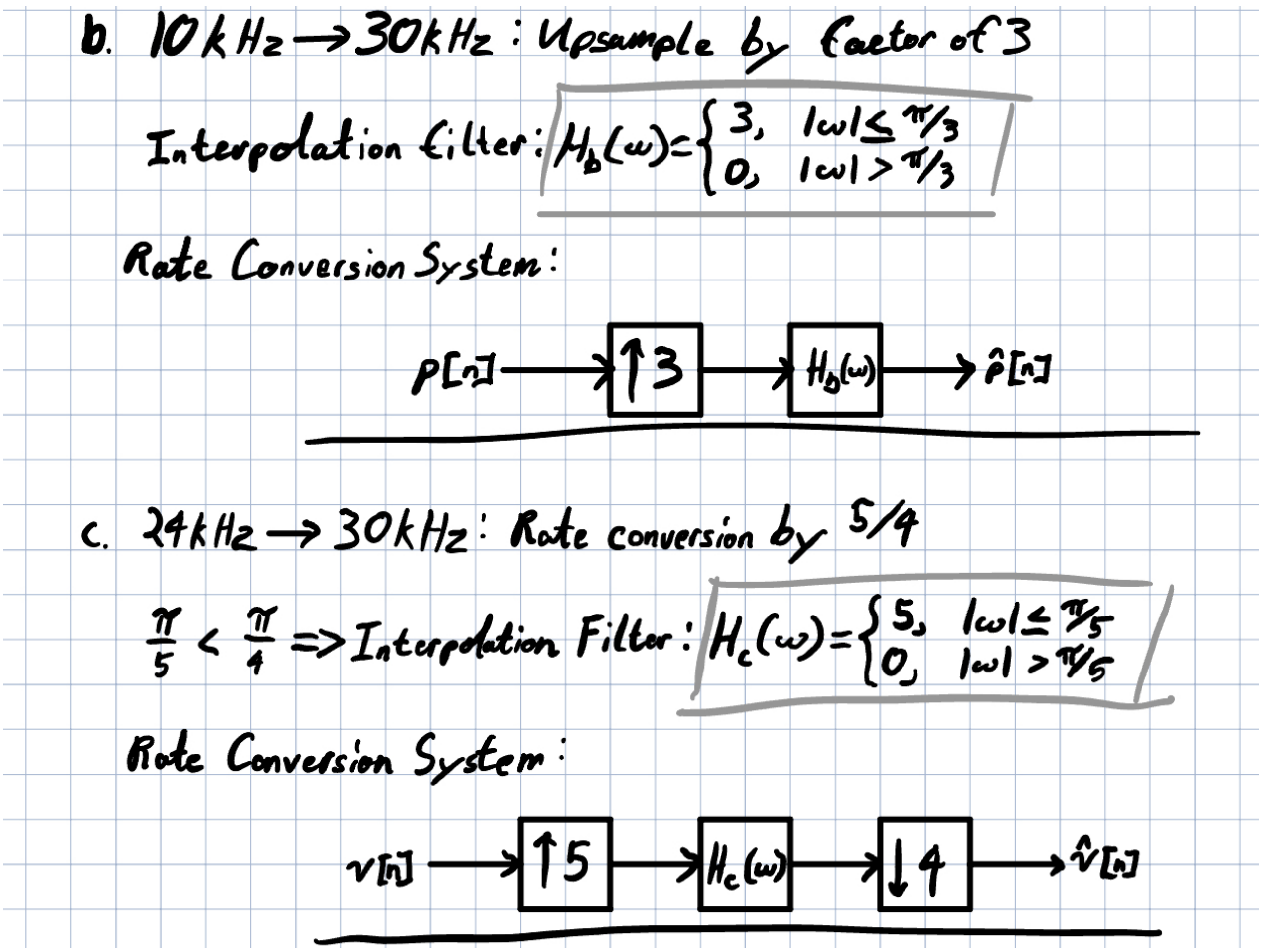

Figure 2: Diagrams for Question 7 (a) & (b)

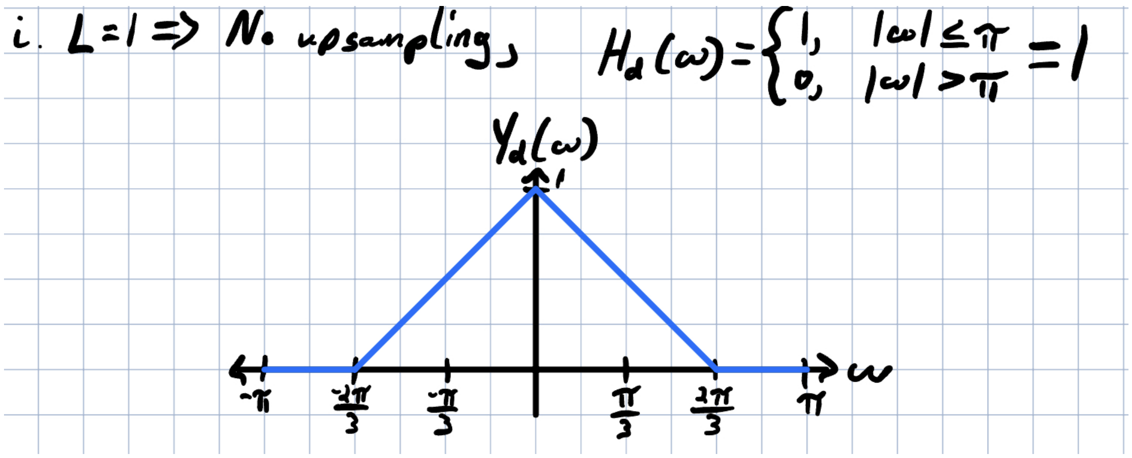

Figure 3: $Y_d(\omega)$, Question 8a.i

(b)

i.

See Figure 5

ii.

To account for the upsampling that occurred, our sampling period for reconstruction must be $T_Z = \frac{T}{L} = \frac{1}{600}$s.

iii.

We can use the same process as in (a).iii, where the spectral width between the central and side copies is our transition bandwidth.

$$\text{Transition Bandwidth} = \frac{\frac{16\pi}{9}}{T_Z} - \frac{\frac{2\pi}{9}}{T_Z} = \frac{3200\pi}{3} - \frac{400\pi}{3} = \frac{2800\pi}{3}$$

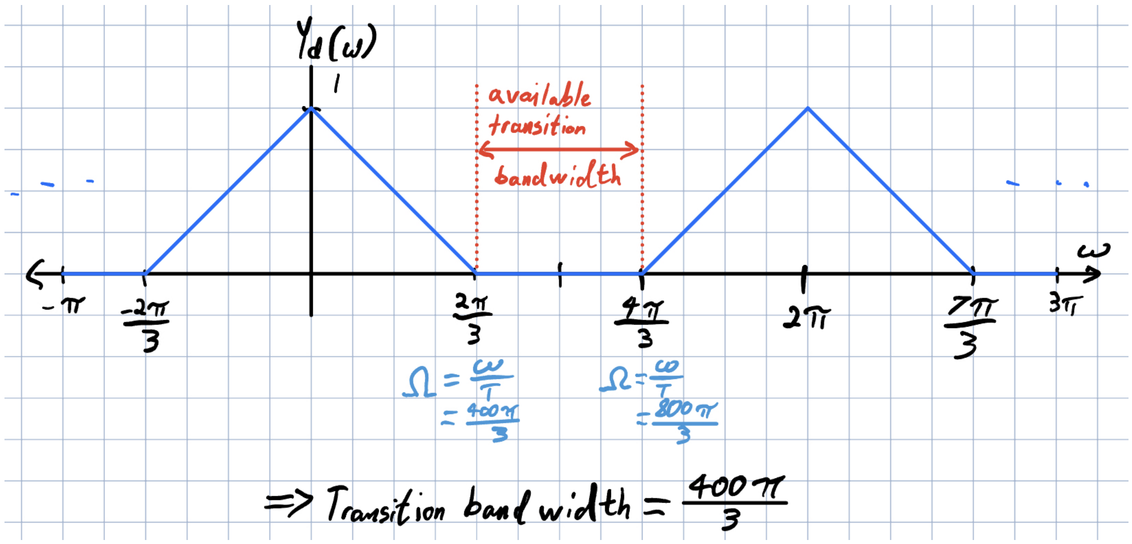

Figure 4: Sketch of Work for Question 8a.ii

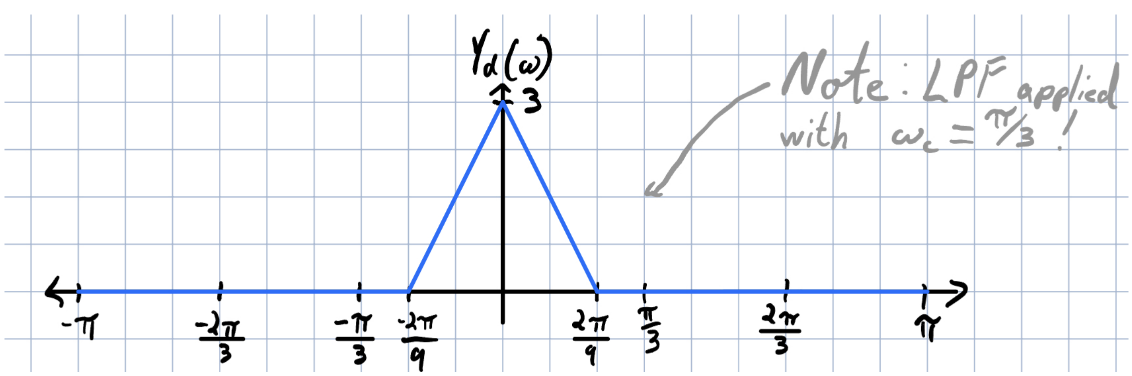

Figure 5: $Y_d(\omega)$, Question 8b.i