October 28, 2022 10:50 AM

# Cond. Lower Bounds from 3SUM (Cont'd)

**Thm 1** (Patrascu '10)

Assuming the Int 3SUM Conj, no $O(n^{3-\delta})$ time alg'm for Zero-Weight Triangle for weighted graphs

Zero-Wt Triangle

$\nearrow$ APSP $\qquad \nwarrow$ 3SUM

OV $\uparrow$ CNF-SAT

## Convolution-3SUM Problem

Given $a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1}, c_0, \dots, c_{n-1}$,
decide $\exists\, i, k$ s.t. $a_i + b_{k-i} = c_k$.
(i.e. $\exists\, i, j$ s.t. $a_i + b_j = c_{i+j}$)

((min,+)-convol: $>$ )

[one-seq vers: $\exists\, i, j,\ a_i + a_j = a_{i+j}$ ].

**Convol-3SUM $\to$ 3SUM : easy**

map $a_i \to$ ~~$(i, a_i)$~~ $\to iM + a_i$

( $(i, a_i) + (j, a_j) = (k, a_k)$ )
$\iff i+j=k$ & $a_i + a_j = a_k$
$\iff a_i + a_j = a_{i+j}$ )
( $iM + a_i + jM + a_j = kM + a_k$
$\iff$ same — — )

**Reduce 3SUM $\to$ Convol-3SUM (for ints)**

( Patrascu '10 / Kopelowitz-Pettie-Porat '16 / C.-He '20)

idea: hashing almost with

idea - hashing almost with the property $h(a) + h(b) = h(a+b)$

i.e. "almost" linear fn

e.g. pick random prime $p \in (R/2, R]$

define $h(x) = x \bmod p$.

**Obs**

(i) $h(a+b) = h(a) + h(b)$ or $h(a) + h(b) - p$

(ii) for any fixed $a, a' \in [U]$ with $a \neq a'$,

$$\Pr[h(a) = h(a')] \le \tilde{O}\left(\frac{1}{R}\right).$$

**Pf of (ii):**

$$\Pr[h(a) = h(a')] = \Pr(a \equiv a' \bmod p)$$

$$= \Pr(\underset{\text{random}}{p} \text{ is a prime divisor of } a - a')$$

$$= \frac{\#\text{ prime divisors of } a - a'}{\#\text{ primes in } (R/2, R)}$$


$$\le O\left(\frac{\log U}{R/\log R}\right) = \tilde{O}\left(\frac{1}{R}\right). \quad \square$$

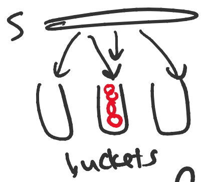
**Cor** for fixed set $S$ of $n$ numbers & fixed $\ell$, the bucket $B_\ell = \{a \in S : h(a) = \ell\}$ has expected size $\tilde{O}\left(\frac{n}{R}\right)$.

To solve 3SUM for set $S$ of $n$ numbers:

choose $R = n$.

call bucket $B_\ell$ good if its size is $\tilde{O}(1)$.

( by Markov's ineq, $\Pr[X > c\,E[X]] \le \frac{1}{c}$ )




(by Markov's ineq, $\Pr[X > cE[X]] \le \frac{1}{c}$)

$\Pr$[ ans is not in good buckets] $< \delta$)

for each nonempty good bucket $B_\ell$,

pick a rand. $x \in B_\ell$

set $x$'s index to $\ell$ (i.e. set $a_\ell = x$)

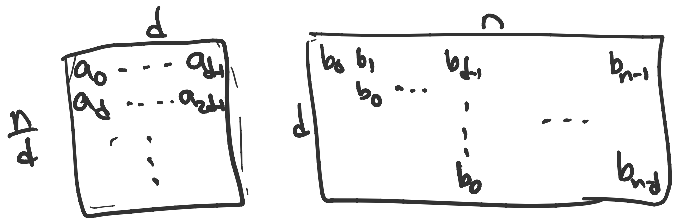
Solve convol-3SUM on $a_1, \dots, a_n$

$\Pr$( ans is found) $\geq \tilde{\Omega}(1)$

repeat $\tilde{O}(1) \times \log n$ times.

## Reduce Convol3SUM → Zero-Weight Triangle:

idea - similar to (min,+)-Convol → (min,+)-MM



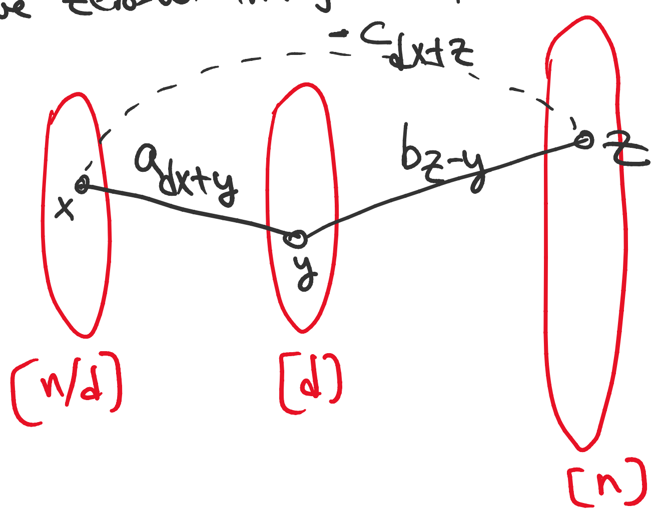

$d = \sqrt{n}$

$M^*(\sqrt{n}, \sqrt{n}, n)$

$= O(\sqrt{n} \cdot M^*(\sqrt{n}))$

To solve Convol-3SUM for $a_0, \dots, a_{n-1}$, $b_0, \dots, b_{n-1}$, $c_0, \dots, c_{n-1}$:

Solve Zero-Wt Triangle on following tripartite graph:




$\exists$ zero-wt tri:

$a_{dx+y} + b_{z-y} = c_{dx+z}$

$\exists\ a_i + b_j = c_{i+j}$

write $i = dx + y$

$z = j + y$

Set $d = \sqrt{n}$

break into $\sqrt{n}$ graphs with $O(\sqrt{n})$ vertices

$$\Rightarrow \text{ time } O(\sqrt{n} \cdot T_{zwt}(\sqrt{n}))$$
$$\leq O(\sqrt{n} \cdot (\sqrt{n})^{3-\delta})$$
$$= O(n^{2-\delta/2}). \quad \square$$