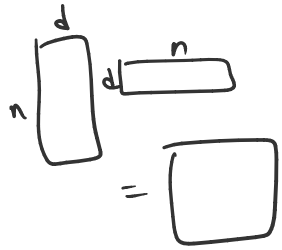
Problem: Boolean Orthogonal Vectors (OV)

Given sets A, B of n vectors in $\{0,1\}^d$,
decide $\exists\, a \in A,\ b \in B$ s.t. $a \cdot b = 0$
$\Updownarrow$
$$\sum_{i=1}^{d} a(i) \cdot b(i) = 0.$$

naive alg'm: $O(dn^2)$
or $O(M(n,d,n))$
$\leq O(d^{\omega-2} n^2)$



Can we beat n^2?
$O(2^d n)$
or $O(n + 4^d)$ ← good only when $d = o(\log n)$

OPEN: $O(n^{2-\delta})$ for $d \gg \log n$??

OV Conjecture No alg'm for OV in $O(d^{O(1)} n^{2-\delta})$ time

Thm (Williams '05)
SETH $\Rightarrow$ OV Conj.

Pf: Reduce f-sparse k-SAT $\to$ OV.

Suppose OV could be solved in $T(n,d) = O(d^{O(1)} n^{2-\delta})$ time. ←

Given f-sparse k-CNF formula F with n vars $x_1, \ldots, x_n$, & $m \leq fn$ clauses $C_1, \ldots, C_m$

For each assignment φ of $x_1, \ldots, x_{n/2}$,
define vector $a_\varphi \in \{0,1\}^m$,
$a_\varphi[j] = 0$ iff C_j satisfied by φ.

define vector a_φ ...

$a_\varphi[j] = 0$ iff C_j satisfied by φ.

For each assignment ψ of $x_{n/2+1}, \ldots, x_n$,

define vector $b_\psi \in \{0,1\}^M$,

$b_\psi[j] = 0$ iff C_j satisfied by ψ.

Solve OV on these 2 sets of $N = 2^{n/2}$ vectors.

Correctness: $\exists$ sat. assignment for F

$\iff \exists \varphi, \psi$, s.t. $\forall j$, $\underbrace{C_j \text{ is satisfied by } \varphi \text{ or by } \psi}_{a_\varphi[j]=0 \text{ or } b_\psi[j]=0}$.

$\iff \exists \varphi, \psi$, $\sum_{j=1}^{m} a_\varphi[j] \cdot b_\psi[j] = 0$

Runtime: $T(2^{n/2}, fn) \leq O\left((fn)^{O(1)} \left(2^{n/2}\right)^{2-\delta+o(1)}\right)$

$\lesssim O^*\left(2^{(1-\delta/2)n}\right)$

contradicting SETH. $\square$

Rmk: generalizes to the k-OV problem:

Given $A_1, \ldots, A_k \subseteq \{0,1\}^d$,

decide $\exists a_1 \in A_1, \ldots, a_k \in A_k$ s.t.

$$\sum_{i=1}^{d} a_1[i] \cdots a_k[i] = 0.$$

Conj no $O(d^{O(1)} n^{k-\delta})$ alg'm for k-OV.

OV → Diameter in Sparse Graphs

naive: $O(mn)$

Thm (Roditty, Vassilevska W. '13)

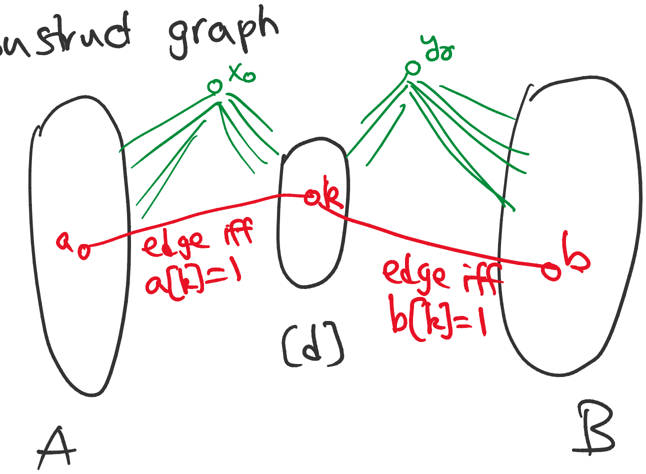
Assuming SETH, no $O(m^{2-\delta})$ alg'm for ... of undir. unweighted graph

Assuming SETH, no $O(m^{[illegible]})$ alg'm to compute diameter of undir. unweighted graph with m edges.

Pf: Suppose there is diam alg'm in $T(m) = O(m^{2-\delta})$ time.

Given sets A, B of n vectors in $\{0,1\}^d$,

Construct graph



Correctness: $d(a,b) = \begin{cases} 2 & \text{if } a\cdot b > 0 \\ \geq 3 & \text{else} \end{cases}$

if no orth pair, diam $= 2$

if $\exists$ orth pair, diam ≥ 3

Runtime: # vertices $O(n+d)$

edges $O(dn)$

$$\Rightarrow \text{OV in time } T(O(dn)) \leq O((dn)^{2-\delta}) = O(d^{O(1)} n^{2-\delta}). \quad \square$$

Further Consequence: assuming SETH, no $O(m^{2-\delta})$ alg'm for $\left(\frac{3}{2}-\varepsilon\right)$-factor approx of diameter

(Rmk: Chechik et al. '14: $\tilde{O}(m^{1.5})$ alg'm for $\frac{3}{2}$-approx.)

More Consequence: assuming SETH,
no $O(m^{2-\delta})$ alg'm for $(2-\varepsilon)$-approx
for S-T diameter

(given $S, T \subseteq V$, compute $\max_{s\in S, t\in T} d(s,t)$).

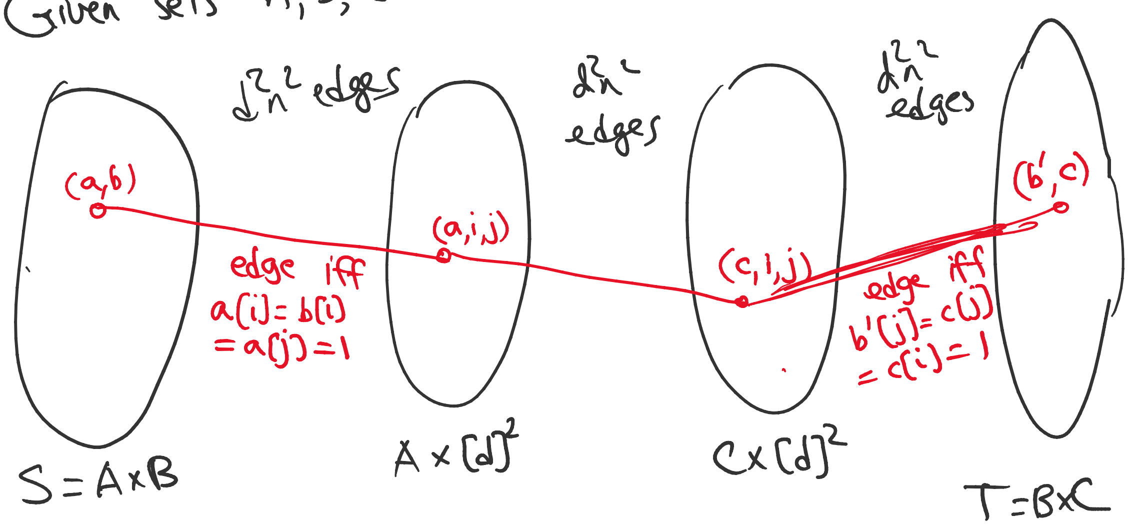
(Pf: skip x_0, y_0)

Thm (Backurs et al. '18) Assuming SETH,
no $O(m^{1.5-\delta})$ alg'm for $(8/5-\varepsilon)$-approx. diam
or for $(7/3-\varepsilon)$-approx S,T diam.

⋮

Pf (for S-T diam): Reduce 3-OV to S-T diam.
Suppose S-T diam could be solved in $T(m) = O(m^{1.5-\delta})$ time.

Given sets A, B, C of n vectors in $\{0,1\}^d$,



Correctness: if no orth triple,
$\forall (a,b)\in S, (b',c)\in T,$
$\exists i,\quad a[i] = b[i] = c[i] = 1.$
$\exists j\quad a[j] = b'[j] = c[j] = 1.$

$\exists i, \quad a[i] =$

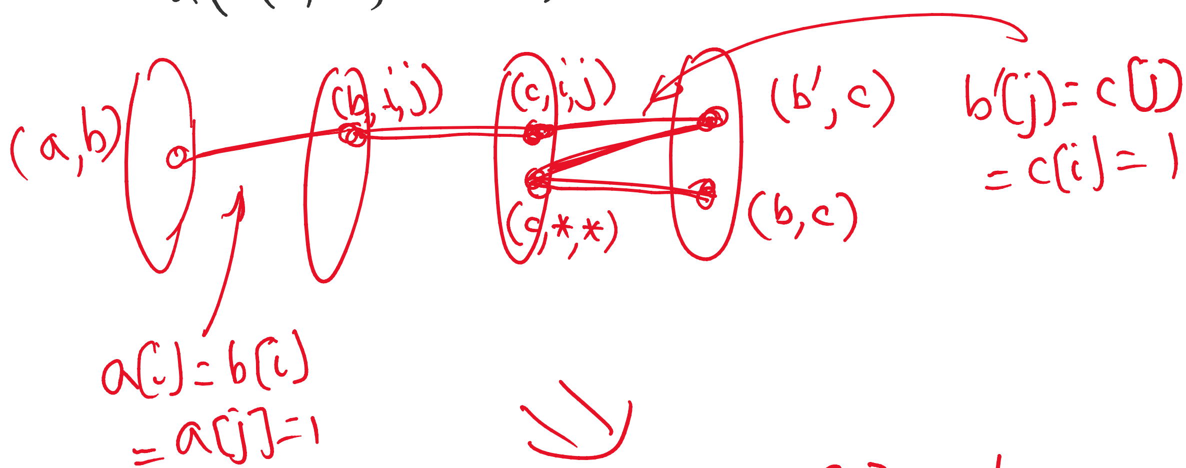
$\exists j, \quad a[j] = b'[j] = c[j] = 1.$

$\Longrightarrow \quad d((a,b),(b',c)) = 3.$

$\Longrightarrow$ S-T diam $= 3$.

if $\exists$ orth triple (a,b,c),

$d((a,b),(b,c)) \cancel{\geq 3} \geq 7$



$\Longrightarrow$

$a[i] = b[i] = c[i] = 1.$

$\Longrightarrow (a,b,c)$ not orth triple.

length 5 can't happen

Runtime:

$\#$ edges $= O(d^2 n^2)$

$\Longrightarrow \quad O\left((d^2 n^2)^{1.5-\delta} \right)$

$= O\left(d^{O(1)} n^{3-2\delta} \right)$

for 3-OV. $\square$