October 7, 2022 2:25 PM

Reduction 4: Superadditivity → Unbounded Knapsack

Assume unbdd knapsack could be solved in $O(t^{2-\delta})$ time.

To solve superadditivity, given $f_0, \dots, f_{n-1}$: increas.

for each $i = 0, \dots, n-1$

Create items (i, f_i), $(t-i, U-f_i)$ (for suff. large $U \geq 2n \max f_i$)

weight, profit (type 1); weight, profit (type 2)

with $t = 2n$ (capacity)

Solve unbdd knapsack in $O(n^{2-\delta})$ time

Claim: f is superadditive $\iff$ max profit $\leq U$.

Pf: ($\Leftarrow$) Suppose f is not superadditive.

Then $\exists i, j$, $f_i + f_j > f_{i+j}$.

Choose 3 items:

(i, f_i), (j, f_j), $(t-i-j, U-f_{i+j})$

profit $f_i + f_j + U - f_{i+j} > U$.

($\Rightarrow$) Suppose f is superadditive.

Consider any feasible sol'n I.

If I uses items (i, f_i), (j, f_j), with $i+j \leq n$

can replace with $(i+j, f_{i+j})$

& profit can only increase.

(note: duplicates are ok for unbdd knapsack)

I can use ~~only~~ exactly one item of type 2, of the form $(t-k, U-f_k)$. (because U is large)

$t-k \geq t-n$

So, $I = \{(i, f_i), (t-k, U-f_k)\}$, so, this implies items of type 1 have total wt $\leq n$

so all items of type 1 can be combined into one by the above replacement idea

with $i + t - k \leq t \Rightarrow i \leq k$

profit $f_i + U - f_k \leq U$. $\square$

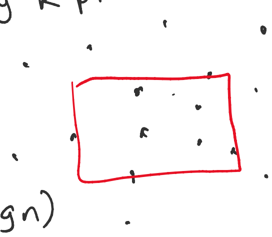
Reduction 5: Unbdd Knapsack → 0/1 Knapsack

idea - any # can be expressed as sum of distinct powers of 2.

To solve unbdd knapsack:
for each item (w_i, p_i),
create new item $(2^\ell w_i, 2^\ell p_i)$
for $\ell = 0, \dots, \log U$.
Solve 0/1 Knapsack on these $O(n \log U)$ items with same t. $\square$

A geometric appl'n:

Problem: Given n pts in 2D and k,
find min area axis-aligned rectangle containing k pts.



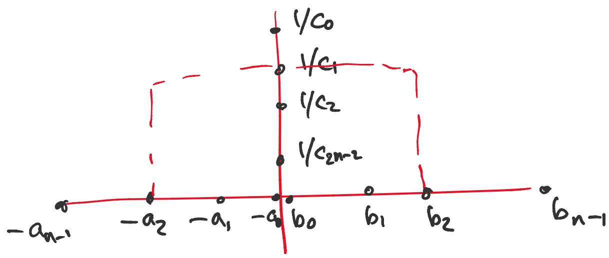
C., Har-Peled '20: $O(n^2 \log n)$

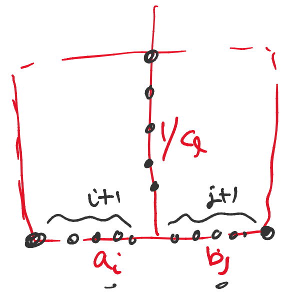
Thm: If min-k-enclos rect could be solved in $O(n^{2-\delta})$ time,
then (min,+)-Convol " " " $O(n^{2-\delta'})$ time.

Pf:
Reduction: Detect-One (min,+)-Convol Decis → min-k-enclos rect.

Given $a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1}, c_0, \dots, c_{2n-2}$:
To decide whether $\exists i, j,\ a_i + b_j \le c_{i+j}$:
w.l.o.g. assume a_i, b_i, c_i increas. & in $[0, U)$.
with $a_0 = 0$, $b_0 = 0$.

Create pts $(-a_i, 0)$ $\quad i = 0, \dots, n-1$
$(0, b_j)$ $\quad j = 0, \dots, n-1$
$(0, \frac{1}{c_\ell})$ $\quad \ell = 0, \dots, 2n-2$
Set $k = 2n+1$.





\# pts enclosed

$= i+1 + j+1 + 2n-2-\ell+1$

$= i+j-\ell+2n+1$

$= k$

$\iff i+j = \ell.$

$$\text{area} = (a_i+b_j)\cdot\frac{1}{c_\ell} \le 1$$

$$\iff a_i + b_j \le c_\ell. \qquad \square$$

Conditional Lower Bds Based on SAT

(CNF-) SAT Problem Given CNF formula F with n vars, decide if $\exists$ satisfying assignment

e.g. $\underbrace{(x_1 \vee x_2 \vee \overline{x_3})}_{\text{clause}} \wedge (\overline{x_2} \vee x_3 \vee x_4) \wedge \cdots$

literal

k-SAT: version for max clause length k