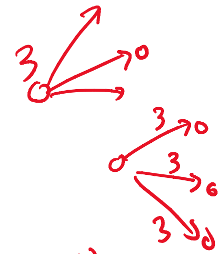
September 28, 2022 10:44 AM

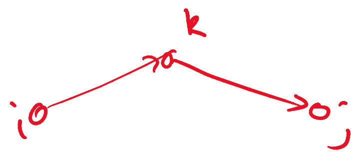
APSP for real vertex wts:



Lemma Given $n \times n$ real matrix A & $n \times n$ matrix B where all entries of B are either 0 or ∞ ("binary")

We can compute (min,+)-MM of A, B in $O(n^{(3+\omega)/2})$ time $\leq O(n^{2.687})$

$$\boxed{\begin{aligned} \underline{c_{ij}} &= \min_k (a_{ik} + b_{kj}) \\ &= \min_{k:\, b_{kj} \neq \infty} \underline{\underline{a_{ik}}} \end{aligned}}$$



Pf: Sort each row of A & map entries to $[n]$.

Divide $[n]$ into r intervals $I_1, \ldots, I_r$ of length n/r

Step 1. for each I_g, compute $d_{ij}^{(g)} = \bigvee_k [a_{ik} \in I_g] \wedge (b_{kj} \neq \infty)$

Boolean MM $\Rightarrow O(r n^{\omega})$

let g_{ij} = smallest g s.t. $d_{ij}^{(g)}$ true

Step 2. Compute $c_{ij} = \min_{\substack{a_{ik} \in I_{g_{ij}} \\ b_{kj} \neq \infty}} a_{ik}$

$\Rightarrow O(n^2 \cdot \frac{n}{r})$

total $O(r n^{\omega} + \frac{n^3}{r}) \Rightarrow O(n^{\frac{3+\omega}{2}})$. □

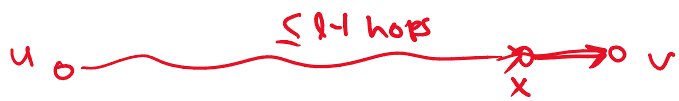
(Small improv: $O(M(rn, n, n) + \frac{n^3}{r})$
$\Rightarrow \underline{\tilde{O}(n^{2.66})}$)

Alg'm (C.'07):

Case 1. Short paths with $\leq L$ hops

idea. don't use repeated squaring
multiply iteratively

Assume we have computed all shortest paths with $\leq \ell-1$ hops.



$$d^{(\ell)}(u,v) = \left(\min_{\substack{x \in V: \\ (x,v) \in E}} d^{(\ell-1)}(u,x) \right) + w(v)$$

Lemma (let $b_{xv} = \begin{cases} 0 & \text{if } (x,v) \in E \\ \infty & \text{else} \end{cases}$

$\Rightarrow \boxed{O(L\, n^{\frac{3+\omega}{2}})}$ time

Case 2. paths with $\geq L$ hops

$\boxed{\tilde{O}\left(\frac{n^3}{L}\right)}$ as before

Total: $\tilde{O}\left(L\, n^{\frac{3+\omega}{2}} + \frac{n^3}{L} \right)$ $\quad L^2 = n^{\frac{3-\omega}{2}}$

Choose $L = n^{\frac{3-\omega}{4}} \Rightarrow \tilde{O}\left(n^{3-\frac{3-\omega}{4}}\right)$
$\leq \boxed{O(n^{2.844})}$.

Open: better?

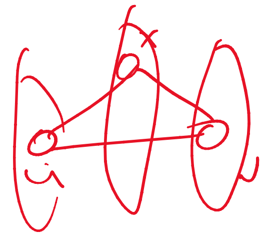
Real-edge weights?

Part III. Conditional Lower Bds

Part II: Conditional Lower Bds

APSP

min-weight triangle ←



min-weight cycle

diameter $\max_{s,t\in V} d(s,t)$

radius $\min_{s\in V} \max_{t\in V} d(s,t)$

s is "center"

⋮

Q: is min-wt triangle, diam, radius, . . . easier than APSP?

APSP Conjecture No $O(n^{3-\delta})$ time alg'm for APSP for arbitrary real edge-weighted graphs for any $\delta>0$.

Int. Version: . . . for arbitrary integer edge-wted graphs in $[U]$

No $\tilde{O}(n^{3-\delta})$ time

$\tilde{O}$ hides polylog(nU) factors.

Goal: prove lower bds for min-wt triangle, radius, . . . conditioned to these conjs.

by reductions

Thm For int wts,

APSP has an $\tilde{O}(n^{3-\delta})$ alg'm for some $\delta>0$

Vassilevska W. & Williams '13 $\iff$ min-wt triangle has an $\tilde{O}(n^{3-\delta'})$ alg'm for some $\delta'>0$

Abboud, Grandoni, Vassilevska W. '15 $\iff$ radius has an $\tilde{O}(n^{3-\delta''})$ alg'm for some $\delta''>0$.

Abboud, Grandoni, Vassilevska W. '15 $(\Leftarrow)$ radius has an $O(\ldots$ for some $\delta'' > 0$.

$(\Rightarrow$ easy

$\Leftarrow$ will reduce APSP to min-wt triangle / radius $)$

Step 0. Reduce APSP $\to$ (min,+)-MM by repeated squaring (or Munro's recursion)

Problem (min,+)-MM-Decision

Given $n \times n$ matrices A, B, D, for each i, j, decide whether $\min_k (a_{ik} + b_{kj}) \leq d_{ij}$

i.e. $\exists k,\ a_{ik} + b_{kj} \leq d_{ij}$.

Step 1. (min,+)-MM $\to$ (min,+)-MM-Decision

(assume ints in $[U]$)

Suppose (min,+)-MM-Decis. could be solved in $T(n)$ time.

Solve (min,+)MM by simultaneous binary search

$\Rightarrow O(T(n) \log U)$. $\square$

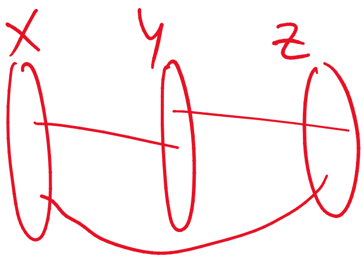
(for real, rand. search ...)

Problem Negative Weight Triangle (NWT)

Given tripartite weighted graph. $G = (X \cup Y \cup Z, E)$

$(E \subseteq (X \times Y) \cup (Y \times Z) \ldots$

Given tripartite weighted graph, $G=(X\cup Y\cup Z, E)$
$(E\subseteq (X\times Y)\cup(Y\times Z)\cup(Z\times X))$



decide $\exists\, x\in X,\ y\in Y,\ z\in Z,$
$w(x,y)+w(y,z)+w(z,x)\le 0.$

Report-One version: if yes, return one such (x,y,z)

All-Edges (AE) version: for every edge $(z,x)\in Z\times X$, decide $\exists\, y\in Y$ s.t.
$w(x,y)+w(y,z)+w(z,x)\le 0.$

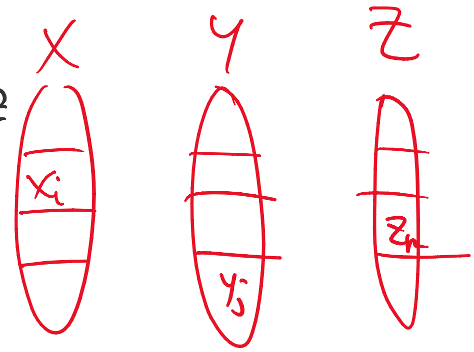
Obs (min) MM-Decis $\equiv$ NWT in AE version

Step 2. Reduce NWT in AE Version $\rightarrow$ NWT in Report-One Version

Suppose Report-One NWT could be solved in $T(n)=\hat{O}(n^{3-\delta})$ time.

To Solve AE-NWT:

Divide X into $X_1,\dots,X_r$ of size $\frac{n}{r}$
Y $Y_1,\dots,Y_r$
Z $Z_1,\dots,Z_r$



for each $i,j,k\in[r]$
repeat {
call oracle to find neg-wt triangle $(x,y,z)\in X_i\times Y_j\times Z_k$ ← $T(\frac{n}{r})$
if exist
set ans for (z,x) to yes

if exist

set ans for (z,x) to yes

remove (z,x) from E

else break

}

Total time: # oracle calls $= O(r^3 + K)$

"yes" edges

$$\Rightarrow O\left((r^3 + K)\, T\left(\frac{n}{r}\right) \right)$$

$$\leq O\left((r^3 + n^2)\, T\left(\frac{n}{r}\right) \right)$$

Set $r = n^{2/3}$ $\Rightarrow O\left(n^2 \cdot T(n^{1/3}) \right)$

$$\leq O\left(n^2 \cdot (n^{1/3})^{3-\delta} \right)$$

$$= O\left(n^{3-\frac{\delta}{3}} \right). \quad \square$$