Sketch of C., V, Xu '21:

Phase I. for each $\ell = (3/2)^i$, in increas order:

compute all shortest paths with $\leq \ell$ hops

from $R_{\delta\ell}$ $\quad$ ($\delta = 0.1$)

& to $R_{\delta\ell}$ $\leftarrow$ $|R_{\delta\ell}| = \tilde{O}(\frac{n}{\ell})$

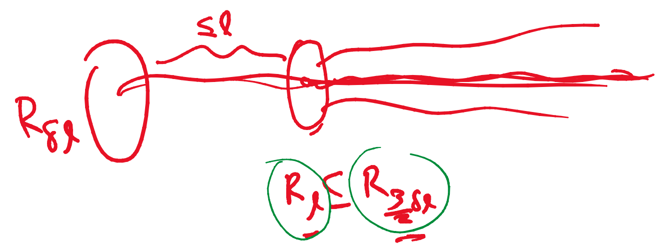
Similar to Zwick

$$\Rightarrow \quad \tilde{O}\left(c\ell \, M\left(\frac{n}{\ell}, \frac{n}{\ell}, n\right)\right)$$

Phase II. for each $\ell = (3/2)^i$ in decreas. order,

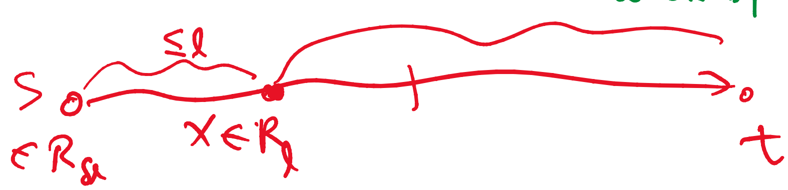
compute all shortest paths from $R_{\delta\ell}$ as follows:

- if $\leq \ell$ hops, already done in Phase I.
- if $> \ell$ hops.



$\sim \frac{n}{\ell} \times \frac{n}{\ell}$ matrix A, entries in $[c\ell]$

$\sim \frac{n}{\ell} \times n$ matrix B, entries could be large but ok by new lemma



$$\tilde{O}\left(c\ell \cdot M\left(\frac{n}{\ell}, \frac{n}{\ell}, n\right) \right)$$

by New Lemma

by "quick upper bd",

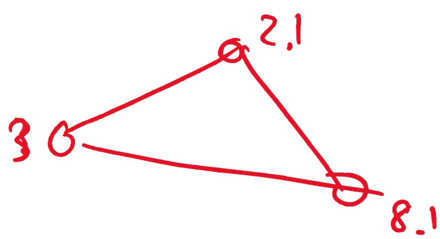
$$\tilde{O}\left(c\ell \cdot \ell \left(\frac{n}{\ell}\right)^{\omega} \right)$$

$$= \tilde{O}\left(c \frac{n^{\omega}}{\ell^{\omega-2}} \right)$$

$\Longrightarrow$ Sum over $\ell = (3/2)^i \Longrightarrow$ $\boxed{\tilde{O}(cn^{\omega})}$

Real-Vertex-Weighted Case

Warm-Up: Min-Weight Triangle for real vertex weights

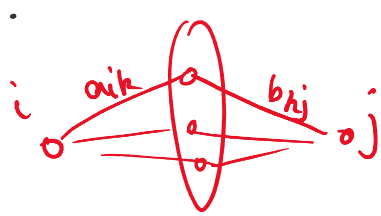


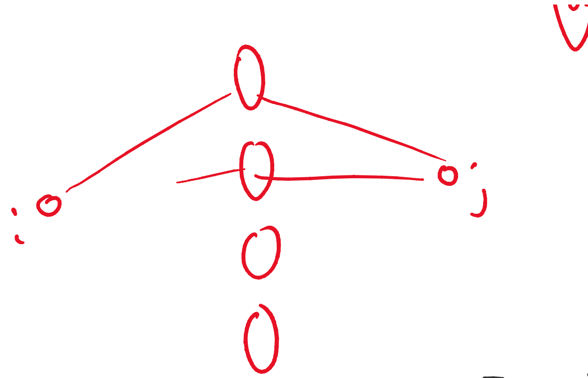
Lemma Given $n \times n$ Boolean matrices A, B, can compute min-witness product C:

$$c_{ij} = \min\{k : a_{ik} \wedge b_{kj} \text{ is true}\}$$

in $O(n^{2.529})$ time.

Pf:





Divide $[n]$ into r intervals $I_1, \ldots, I_r$ of length n/r.

Step 1. for each I_g $(g \in \{1, \ldots, r\})$

compute $d_{ij}^{(g)} = \bigvee_{k \in I_g} (a_{ik} \wedge b_{kj})$ ← rect. Bool. MM

let g_{ij} be smallest g s.t. $d_{ij}^{(g)}$ true.

$$\Rightarrow \quad O\left(r \cdot M\left(n, \frac{n}{r}, n\right)\right)$$

Step 2. compute $c_{ij} = \bigvee_{k \in I_{g_{ij}}} (a_{ik} \wedge b_{kj})$

by brute force

$$\Rightarrow \quad O\left(n^2 \cdot \frac{n}{r}\right)$$

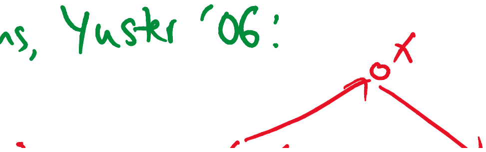
total

$$\Rightarrow \quad O\left(r \cdot M\left(n, \frac{n}{r}, n\right) + \frac{n^3}{r}\right)$$

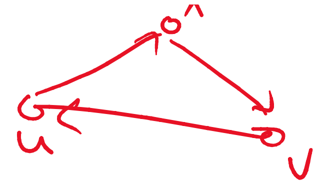
like Zwick!

$$\Rightarrow \quad O(n^{2.529}) \quad . \quad \square$$

Algm for min-wt triangle for vertex wt:
by Vassilevska, Williams, Yuster '06:



by Vassilevska, ...



for each u, v,

compute $c_{uv} = \min\{ w(x) : (u,x) \in E \wedge (x,v) \in E \}$

min-witness product after sorting x by weight

$$\min_{u,v} \left(c_{\cancel{uv}\,vu} + w(u) + w(v) \right)$$

$\Rightarrow \boxed{O(n^{2.529})}$ time

Czumaj-Lingas '90: $\tilde{O}(n^\omega)$

by a clever recursive alg'm

APSP for real vertex wts:

Lemma Given $n \times n$ real matrix A

& $n \times n$ matrix B

where all entries of B are either 0 or ∞ ("binary")

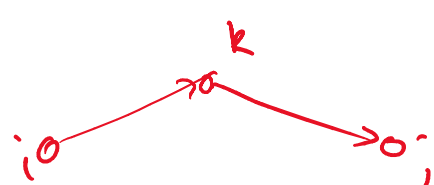
We can compute $(\min,+)$-MM of A, B

in $O(n^{(3+\omega)/2})$ time

$\leq O(n^{2.687})$

$$c_{ij} = \min_k (a_{ik} + b_{kj})$$

$$= \min_{k : b_{kj} \neq \infty} a_{ik}$$



Pf: Sort each row of A & map entries to $[n]$.

Divide $[n]$ into r intervals $I_1, \ldots, I_r$ of length n/r

Step 1. for each I_g,

compute $d_{ij}^{(g)} = \bigvee_k [a_{ik} \in I_g] \wedge (b_{kj} \neq \infty)$

Boolean MM $\Rightarrow O(r n^{\omega})$

let g_{ij} = smallest g s.t. $d_{ij}^{(g)}$ true

Step 2. compute $c_{ij} = \min\limits_{\substack{a_{ik} \in I_{g_{ij}} \\ b_{kj} \neq \infty}} a_{ik}$

$\Rightarrow O(n^2 \cdot \frac{n}{r})$

total $O(r n^{\omega} + \frac{n^3}{r}) \Rightarrow O(n^{\frac{3+\omega}{2}})$. $\square$

(Small improv: $O(M(rn, n, n) + \frac{n^3}{r})$

$\Rightarrow \tilde{O}(n^{2.66})$)