most signif. digit position $\nearrow$ $\quad\quad \max_k (a_{ik}+b_{kj})$

Choose $M = n+1$.

Runtime: $O(n^\omega)$ arith ops on numbers in $[M^c]$ $\leftarrow$ $O(c \log M)$ bits

each arith op takes $\tilde{O}(c \log M) = \tilde{O}(c)$ time by FFT.

$\Rightarrow$ total $\tilde{O}(cn^\omega)$. $\square$

issue - when reducing APSP to $(\min,+)$-MM, entry values get bigger $\rightarrow$ $[c]$

$\Rightarrow$ $\tilde{O}(cn^{\omega+1})$ bad!

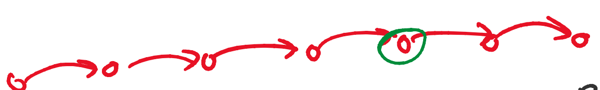
first idea - short vs. long paths

Case 1. short paths with $\leq L$ edges/hops.

repeated squaring to $O(\log n)$ $(\min,+)$-MM in $[cL]$

$\Rightarrow$ $\tilde{O}(cLn^\omega)$

Case 2. long paths with $\geq L$ edges/hops.



Hitting Set Lemma $\quad$ $\exists$ subset $R \subseteq V$ of size $|R| = O\left(\frac{n}{L}\log n\right)$

that hits all shortest paths with $\geq L$ edges.

Pf: take a random subset R with sampling prob. $p = \frac{3}{L}\log n$

Fix s,t. ... the shortest path

Fix s,t.
Let π_{st} be vertices in the shortest path from s to t.

Suppose $|\pi_{st}| \geq L$.

$$\Pr[R \text{ does not hit } \pi_{st}] = (1-p)^{|\pi_{st}|}$$
$$\leq (1-p)^L$$
$$\leq e^{-pL}$$
$$\leq e^{-3\log n}$$
$$= \frac{1}{n^3}.$$

$1-x \leq e^{-x}$

$$\Rightarrow \Pr\left[\bigvee_{s,t} (R \text{ does not hit } \pi_{st})\right]$$

$$\underset{\text{by union bd}}{\leq} n^2 \cdot \frac{1}{n^3} = \frac{1}{n}. \qquad \square$$

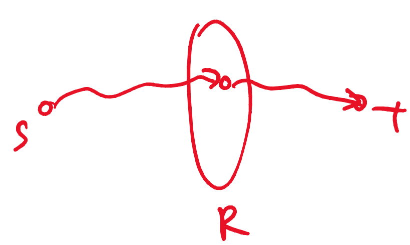
$\Pr(A \cup B) \leq \Pr(A) + \Pr(B)$

"probabilistic method"

[Alternative Pf: by greedy alg'm for hitting set]

Run SSSP (single-source) from each $r \in R$ $\leftarrow$ $\tilde{O}(|R|m)$ time by Dijkstra

& to each $r \in R$



for each $s,t \in V$,

$$\min_{r \in R} \left(d(s,r) + d(r,t) \right) \Leftarrow O(|R|n^2) \text{ time}$$

$$\Rightarrow \tilde{O}\left(\frac{n^3}{L}\right) \text{ time.}$$

Total time: $\tilde{O}\left(\underline{cLn^{\omega}} + \frac{n^3}{L}\right)$

Choose $L = \frac{n^{\frac{3-\omega}{2}}}{\sqrt{c}} \Rightarrow \boxed{\tilde{O}\left(\sqrt{c}\, n^{\frac{3+\omega}{2}}\right)}$

$L^2 = \frac{n^{3-\omega}}{c}$

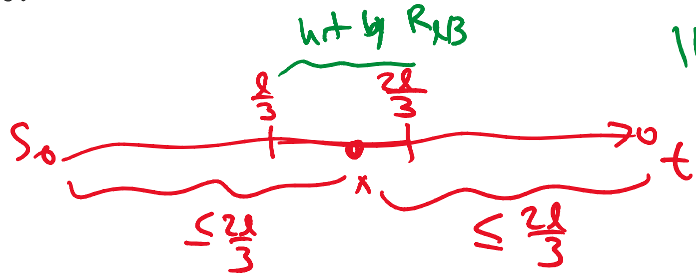
$\leq O(n^{2.687})$ if $c = O(1)$

Zwick's better idea - use hitting set also for short case!

Suppose we have computed all shortest paths with $\leq \frac{2\ell}{3}$ hops.

If shortest path from s to t has # hops $\in \left(\frac{2\ell}{3}, \ell\right]$,

$|R_{\ell/3}| = \tilde{O}\left(\frac{n}{\ell}\right)$



compute $\min\limits_{x \in R_{\ell/3}} \left(d(s,x) + d(x,t)\right)$

$\Rightarrow$ (min,+)-MM of $n \times |R_{\ell/3}|$ matrix & $|R_{\ell/3}| \times n$ matrix

rectangular matrices

$\Rightarrow$ time $\tilde{O}\left(c\ell \cdot M(n, \frac{n}{\ell}, n)\right)$

by Lemma

T ... $\ell = \left(\frac{3}{2}\right)^i \leq L$ $M(n,n,n)$

by Lemma

Try all $\ell = \left(\frac{3}{2}\right)^i \leq L$

$M(n,n,n)$

Total time $\tilde{O}\Big(\underbrace{cL \cdot M(n, \frac{n}{L}, n)}_{\text{short case}} + \underbrace{\frac{n^3}{L}}_{\text{long case}}\Big)$

By quick upper bd;

$$\tilde{O}\left(cL \cdot L^2 \cdot \left(\frac{n}{L}\right)^{\omega} + \frac{n^3}{L}\right)$$

$$= \tilde{O}\left(c \cdot L^{3-\omega} \cdot n^{\omega} + \frac{n^3}{L}\right)$$

If $c = O(1)$, choose $L = n^{\frac{3-\omega}{4-\omega}}$

$$\Rightarrow \tilde{O}\left(n^{3 - \frac{3-\omega}{4-\omega}}\right) \leq O(n^{2.615})$$

By current rect. MM bds, choose $L = n^{0.471}$

if $c = O(1)$, $\boxed{O(n^{2.529})}$

Cond. LB? (C.-Vassilevska W.-Xu '21)

Undirected Case

Seidel '95: $\tilde{O}(n^{\omega})$ for unweighted

Galil-Margalit '97: $\tilde{O}(c^{\frac{\omega+1}{2}} n^{\omega})$ for weights in $[c]$

Shoshan-Zwick '99: $\tilde{O}(c n^{\omega})$

→ C.-V.-Xu '21: $\tilde{O}(c n^{\omega})$ simpler

Obs: $d(\cdot,\cdot)$ is a metric

$$d(u,v) = d(v,u)$$

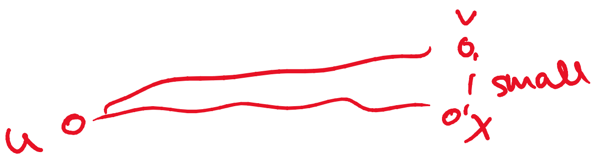
$$d(u,v) \leq d(u,x) + d(x,v) \quad (\Delta \text{ ineq})$$

$$d(u,v) = d(v,u)$$

$$d(u,v) \le d(u,x) + d(x,v) \quad (\Delta \text{ ineq})$$

$$\Rightarrow \quad |d(u,v) - d(u,x)| \le d(x,v)$$

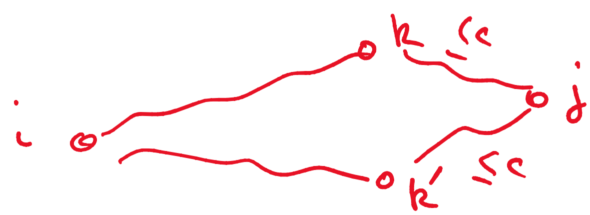
("Δ diff ineq.")



New Lemma Given $n \times n$ matrix A, B where all entries of B are in $[c] \cup \{\infty\}$ but entries of A are arb. ints, s.t. $\forall i,j,k,k'$ with $b_{kj}, b_{k'j} \neq \infty$,

$$|a_{ik} - a_{ik'}| \le 2c$$

We can compute $(\min,+)$-MM of A, B in $\tilde{O}(cn^{\omega})$ time.



Pf: Let $A' = A \bmod 8c$

Compute $(\min,+)$-MM of A' & B $\leftarrow \tilde{O}(cn^{\omega})$ time

Fix i,j.

Pick any k_0 with $b_{k_0 j} \neq \infty$.

Say $a_{ik_0} \bmod 8c \in [3c, 4c]$

$\Rightarrow a_{ik} \bmod 8c \in (c, 6c)$

$\Rightarrow (a_{ik} + b_{kj}) \bmod 8c \in [c, 7c]$ $\forall k$ with $b_{kj} \neq \infty$

$$(a_{ik} + b_{kj}) \operatorname{div} 8c = a_{ik_0} \operatorname{div} 8c.$$

$\square$