Sparse Matrix Mult.

for $n \times n$ matrices A, B with m nonzero entries $(m \ll n^2)$

trivial alg'm: $O(mn)$ time $\quad$ (better than n^ω if $m \ll n^{\omega-1}$)

Yuster & Zwick '05:

idea: high-low trick

let $\deg(k) = |\{i : a_{ik} \neq 0\}|$ $\quad$ $\left(\sum_k \deg(k) \leq m\right)$

let $H = \{k : \deg(k) > \Delta\}$ $\quad$ $|H| \leq \frac{m}{\Delta}$

$L = \{k : \deg(k) \leq \Delta\}$

Low case: compute $c_{ij}^L = \sum_{k \in L} a_{ik} b_{kj}$

$\Rightarrow$ $O(m\Delta)$ time

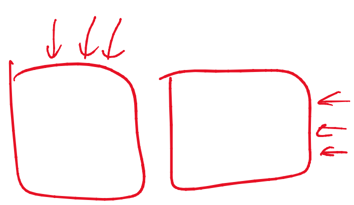
for all kj with $b_{kj} \neq 0$ $\leftarrow$ $O(m)$

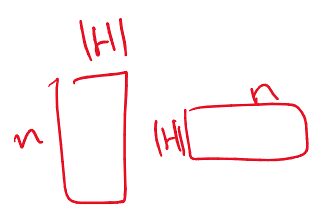
for all i with $a_{ik} \neq 0$ $\leftarrow$ $O(\Delta)$

add $a_{ik} b_{kj}$ to current c_{ij}^L

High case: compute $c_{ij}^H = \sum_{k \in H} a_{ik} b_{kj}$

$\Rightarrow$ by rect. MM, $\leq M(n, \frac{m}{\Delta}, n)$ time





total time $\boxed{O\left(\min_{\Delta}\left(m\Delta + M\left(n, \frac{m}{\Delta}, n\right)\right)\right)}$

e.g. if $m \le n^{1+\alpha/2}$, choose $\Delta = n^{1-\alpha/2}$

$n^{1.156}$

$$\Rightarrow O\left(n^2 + \underbrace{M(n, n^{\alpha}, n)}\right)$$

$$= O(n^{2+\varepsilon}).$$

Appl'n 0: problems about matrices, linear algebra,

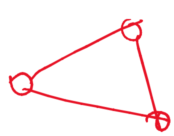
e.g. matrix inverse, $Ax = b$, determinant, ...

Appl'n 1: triangle finding in graphs

Given dir. graph $G = (V, E)$, $|V| = n$, $|E| = m$

decide $\exists$ triangle i.e. $u, x, v \in V$

s.t. $(u,x), (x,v), (v,u) \in E$.



trivial: $O(n^3)$ time

Better? let $a_{uv} = \begin{cases} 1 & \text{if } (u,v) \in E \\ 0 & \text{else} \end{cases}$ (adj matrix)

for each u, v,

$$c_{uv} = \bigvee_{x \in V} (a_{ux} \wedge a_{xv})$$ $\leftarrow$ Boolean MM

reduces to standard MM

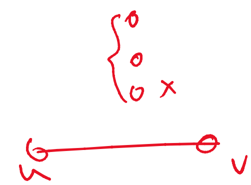
$x \in V$ — reduces to standard MM

$\sum_{x \in U} a_{ux} \cdot a_{xv}$

check whether $c_{uv} = 1 \wedge (v,u) \in E$

$\Rightarrow$ $O(n^{\omega})$ time

$\leq O(n^{2.373})$



Sparse case?

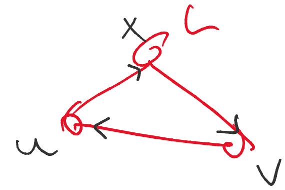
trivial: $O(mn)$

Alon, Yuster, & Zwick's Alg'm ('97)

idea: high-low again!

let $H = \{ v \in V : \deg(v) > \Delta \}$

$L = \{ v \in V : \deg(v) \leq \Delta \}$



Case 1. Some vertex in triangle is in L, say x.

for each $(u,x) \in E$ ← $O(m)$

for each out-neighbor v of x ← $O(\Delta)$

check $(v,u) \in E$

$\Rightarrow$ $O(m\Delta)$ time

Case 2. All 3 vertices in triangle are in H.

$|H| \leq \frac{m}{\Delta}$.

run dense-graph alg'm

$\Rightarrow$ $O\left(\left(\frac{m}{\Delta}\right)^{\omega}\right)$ time

$\ldots (m)^{\omega}$

$\Rightarrow O((\Delta) \dots$

$m\Delta = \left(\frac{m}{\Delta}\right)^{\omega}$

$\Delta^{\omega+1} = m^{\omega-1}$

Total: $O\left(m\Delta + \left(\frac{m}{\Delta}\right)^{\omega} \right)$

choose $\Delta = m^{\frac{\omega-1}{\omega+1}}$

$1 + \frac{\omega-1}{\omega+1} = \frac{2\omega}{\omega+1}$

$$\Rightarrow \boxed{O\left(m^{\frac{2\omega}{\omega+1}}\right)} \lesssim O(m^{1.41})$$

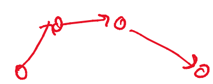
(without MM, $\omega = 3 \Rightarrow O(m^{3/2})$)

(if $\omega = 2$, $O(m^{4/3})$)

[cond. LB: later...]

Extensions:

1. k-cycle (const k)



Alon, Yuster, Zwick: $O(n^{\omega})$ time

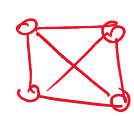
(Color coding)

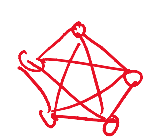
Sparse case: messier

$k=4$: $O(m^{1.48})$

$k=5$: ⋮

2. k-clique (const k)





if k is div. by 3,

define $G' = (V', E')$

$V' = \{$ all $(k/3)$-clique in $G \}$

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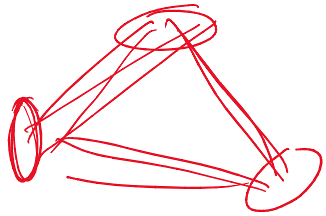
where $V' = \{$ all $(k/3)$-clique in $G \}$

$E' = \{ AB :$ $A \cup B$ is a $\frac{2k}{3}$-clique$\}$

$O(n^{k/3})$



$\Rightarrow$ reduces to triangle finding in G'

$$\Rightarrow \boxed{O\left(\left(n^{k/3}\right)^{\omega}\right)}$$

$\leq O(n^{0.792k})$ time

3. weighted graphs?

min-weight triangle

for each u, v,

compute $c_{uv} = \min_{x \in V} (a_{ux} + a_{xv})$

$\Rightarrow$ (min,+) - matrix mult.

(Strassen ... doesn't work!)

Major Open Q: could (min,+)-MM be solved in $O(n^{3-\varepsilon})$ time?

(sparse case: $O(m^{3/2})$ time)