CS 598TMC, Fall 2022
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Homework 4 (due Dec 1 Thursday 5pm)

Instructions: see previous homework. Only 2 problems this time.

1. [*65 pts*]

   (a) [*35 pts*] Recall the *set intersection query* problem: build a data structure for a collection of sets $S_1, \ldots, S_\ell \subseteq [N]$ with total size $M = \sum_i |S_i|$, so that given any i and j, we can quickly enumerate all elements in $S_i \cap S_j$. In class (on Patrascu's result), we have shown that if there is a data structure that could answer $\widetilde{O}(n^{3/2})$ set intersection queries with total output size $\widetilde{O}(n^{3/2})$ for an input with $M = \widetilde{O}(n^{3/2})$ and $N = \widetilde{O}(n)$ in $\widetilde{O}(n^{2-\delta})$ time for some constant $\delta > 0$, then integer 3SUM could be solved in $\widetilde{O}(n^{2-\delta'})$ time.

   Now consider the *set disjointness query* problem: build a data structure for sets $S_1, \ldots, S_\ell \subseteq [N]$ with total size $M = \sum_i |S_i|$, so that given any i and j, we can quickly decide whether $S_i \cap S_j = \emptyset$. Show that if there is a data structure that could answer $\widetilde{O}(n^{3/2})$ set disjointness queries for an input with $M = \widetilde{O}(n^{3/2})$ and $N = \widetilde{O}(n)$ in $\widetilde{O}(n^{2-\delta})$ time for some constant $\delta > 0$, then integer 3SUM could be solved in $\widetilde{O}(n^{2-\delta'})$ time for some constant $\delta' > 0$.

   Hint: create new sets $S_i \cap [0, N/2)$, $S_i \cap [N/2, N)$, $S_i \cap [0, N/4)$, etc. Queries may be given online.

   (b) [*15 pts*] Consider a data structure version of the *point-rectangle-counting* problem from Homework 3: We are given a set P of m points in 2D, where each p has a weight $w(p)$. We want to build a data structure to answer *rectangle-counting queries*, namely, given query axis-aligned rectangle R, count the number of *distinct* weights among the points of P that are inside R.

   Assuming the 3SUM conjecture, prove that there is no data structure that has $O(m^{1/3-\delta})$ query time and $O(m^{4/3-\delta})$ preprocessing time for any constant $\delta > 0$. (In some sense, this is better than the conditional lower bound from Homework 3, which is trivial if $\omega = 2$.)

   (c) [*15 pts*] Consider the following problem called *dynamic strong connectedness*: decide whether a directed graph with m edges is strongly connected (i.e., for every two vertices u and v, there exists a path from u to v and a path from v to u), under insertions and deletions of edges.

   Assuming the 3SUM conjecture, prove that there is no data structure for dynamic strong connectedness that has $O(m^{1/3-\delta})$ time per edge insertion and edge deletion for any constant $\delta > 0$.

2. [*35 pts*] Consider the following variant of the 3-point collinearity problem: given a sequence of n points $p_1, \ldots, p_n$ in two dimensions, decide whether there exist i and j such that p_i, p_j, p_{i+j} lie on a common line.

   Assuming that the points have integer coordinates, describe a (randomized) algorithm that solves the problem in slightly subquadratic time. Aim for near $O(n^2/\log^2 n)$ time, ignoring $\log\log n$ factors.

   Hint: modify the algorithm from class for integer convolution-3SUM. Note that three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are collinear iff $(x_2 - x_1)(y_3 - y_1) = (x_3 - x_1)(y_2 - y_1)$.