Dynamic Convex Hull in 2D

Static: $O(n \log n)$ time

* e.g. Graham's scan

Dynamic?

- Insert pt
- Delete pt
- Query (e.g. find extreme pt along dir)

$O(\log n)$ time?

Difficulty - each insert/delete can cause $\Omega(n)$ changes to CH.

Suffice to consider upper hull (UH)

by duality: equiv. to lower envelope of lines

Apply - compute $k$-level of $n$ lines

If $k$-level has $h$ vertices,

- $O(h)$ insert/delete queries
- $O((k+h) \log n)$ time alg.?
Warm-Up: Insert-Only

- Insert reduces to finding 2 tangents
  \[ O(\log n) \text{ time by binary search} \]
  \[ \{ \text{split/concatenate lists} \} \quad O(\log n) \]
  \[ \Rightarrow \quad O(\log n) \text{ time} \]

Overmars - van Leeuwen '80: Fully dynamic

**Lemma**

Given 2 vertically separated UHs A, B, we can find the bridge between A, B in \[ O(\log n) \text{ time} \]

**Pf:**
- Dual of bridge
- \[ x = 0 \]

\[ M_B \quad M_A \]
by careful binary search

Case 1. $\times_{mb}$

Case 2. $\times_{mb}$

Case 3. $\times_{mb}$

Case 4. $\times_{ma}$

$\Rightarrow \leq 2 \log n$ iterations.

idea: hull tree

divide by median $x$

recursive on left & right

store bridge

$\Rightarrow O(n)$ space

extreme-pt query: $O(\log n)$ time

insert/delete($p$): if $p$ is on left side
insert/delete on left subtree
Insert/delete($p$):

- If $p$ is an root,
  - Insert/delete on left subtree
  - Delete on right

$\Rightarrow \quad \boxed{O(\log^2 n)}$ time

Issue - balancing?

Option 1 - weight balance

- If left subtree size $\Rightarrow$ c. right subtree size
- Rebuild subtree

$\Rightarrow$ amortization

Option 2 - rotation

AVL/Red-black tree - $O(1)$ rotations per insert/delete

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Improvements:

Chatelle '83: $O(\log n)$ update time but delete-only (null tree w. linear search & amort.)

Fully dynamic

C.'99: $O(\log^2 n)$ amort. update, $O(\log n)$ extract/query

Brodal-Jacobsen '02: $O(\log n)$ amort. update, $O(\log n)$ extract/query
idea. Start w. delete-only apply the logarithmic method to support insert

query time increases to \( O(\log^2 n) \)

Open Qs:

- dynamic convex hull area in 2D
  \( O(\log n) \) update

- dynamic convex hull in 3D
  \( O(\log^4 n) \) update,
  \( O(\log^2 n) \) extreme-pt query [C.'19]

- dynamic convex hull volume in 3D
  \( \widetilde{O}\left(n^{11/12}\right) \)