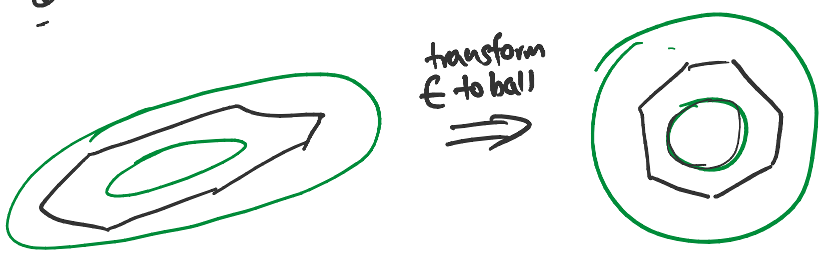
$$= p^{\top} M v$$
$$= p \cdot (M^{\top} v). \quad \square$$

Pf of Lem 1:

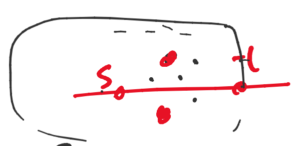
John Ellipsoid (1948)

$\exists$ ellipsoid E s.t.

$$\tfrac{1}{d} E \subseteq CH(P) \subseteq E$$



Pf 2 of Lem 1: (Barequet - Har-Peled '97)



transform$_d$(P):

1. s = any pt of P
2. t = farthest pt from s
3. rotate & scale s.t. $s = (0,0,..,0)$, $t = (1,0,..,0)$
4. project each $p \in P$ to $\hat{p}$ along 1st coord.
5. transform$_{d-1}$($\{\hat{p} : p \in P\}$).

$O(n)$ time

Justification:

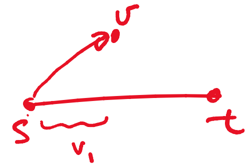
at the end, $P \subseteq [-1,1]^d$.

Claim $\forall$ unit vector v, $w_v(P) \geq \frac{1}{4^d}$.

Pf: Let $v = (v_1, \ldots, v_d)$.

$w_v(P) = \max_{p,q \in P} |(p-q)\cdot v|$

Case 1. $|v_1| \geq \frac{1}{4^d}$.



$$\Rightarrow w_v(P) \geq |(t-s)\cdot v|$$
$$\geq |(t_1 - s_1)\cdot v_1| = |v_1|$$
$$\geq \frac{1}{4^d}.$$

Case 2. $|v_1| < \frac{1}{4^d}$.

By induction, $\exists\, p, q \in P$,

$$(\hat{p}-\hat{q})\cdot\hat{v} \geq \frac{1}{4^{d-1}}\|\hat{v}\|$$

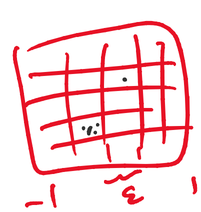
$\|v\| = \sqrt{\|\hat{v}\|^2 + v_1^2} \leq \|\hat{v}\| + |v_1|$

$$\geq \frac{1}{4^{d-1}}\Big(\underbrace{\|v\|}_{=1} - \underbrace{|v_1|}_{< \frac{1}{4^d}}\Big)$$
$$= \frac{1}{4^{d-1}}\underbrace{\left(1 - \frac{1}{4^d}\right)}_{\geq \frac{3}{4}}$$
$$\geq \frac{3}{4^d}.$$

$$\Rightarrow w_v(P) \geq (p-q)\cdot v$$
$$= (\hat{p}-\hat{q})\cdot\hat{v} + (\overbrace{p_1 - q_1}^{\in [-2,2]})\cdot \overbrace{v_1}^{\in (-\frac{1}{4^d}, \frac{1}{4^d})}$$
$$\geq \frac{3}{4^d} - \frac{2}{4^d}$$
$$= \frac{1}{4^d}. \qquad \square$$

ε-Kernel Alg'm 1:

idea - "rounding" pts



1. make P fat in $[-1,1]^d$

$O(n)$ 2. Form uniform grid of side length ε

1. make P fat in $[-1,1]^d$ — $-1 \quad \varepsilon \quad 1$

$O(n)$ time → 2. form uniform grid of side length ε ← $O\left(\frac{1}{\varepsilon^d}\right)$ cells

3. for each grid cell,

$O(n)$ time → put one pt in S

$\Rightarrow$ size $\boxed{O\left(\frac{1}{\varepsilon^d}\right)}$ → $\frac{1}{\varepsilon^{d-1}}$

runtime $\boxed{O\left(n + \frac{1}{\varepsilon^d}\right)}$

Correctness: $\forall$ unit vector v,

$$w_v(S) \geq w_v(P) - 2\sqrt{d}\,\varepsilon$$

$$\geq (1 - O(\varepsilon))\, w_v(P).$$

because $w_v(P) \geq \Omega(1)$ by fatness

$\varepsilon w_v(P)$

$\geq \Omega(\varepsilon)$

$\Leftrightarrow \varepsilon \leq O(\varepsilon w_v(P))$

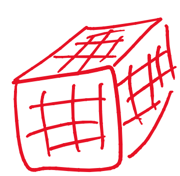
ε-Kernel Alg'm 2:

idea - rounding dirs

1. make P fat in $[-1,1]^d$
2. form set V_δ of $O\left(\frac{1}{\delta^{d-1}}\right)$ dirs with $\delta = \Theta(\varepsilon)$
3. for each $v \in V_\delta$

   put extreme pt p_v maximizing $p_v \cdot v$, q_v minimizing $q_v \cdot v$

   in S



$\Rightarrow$ size $\boxed{O\left(\frac{1}{\varepsilon^{d-1}}\right)}$

runtime $\boxed{O\left(\frac{1}{\varepsilon^{d-1}}\, n\right)}$

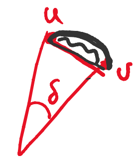
Correctness: $\forall$ unit vector u,

say $w_u(P) = (p-q)\cdot u$ $\quad (p,q \in P)$

let $v \in V_\delta$ s.t. $\angle(u,v) \le \delta$

normalize to unit vec.

$\Rightarrow \|u-v\| \le O(\delta)$

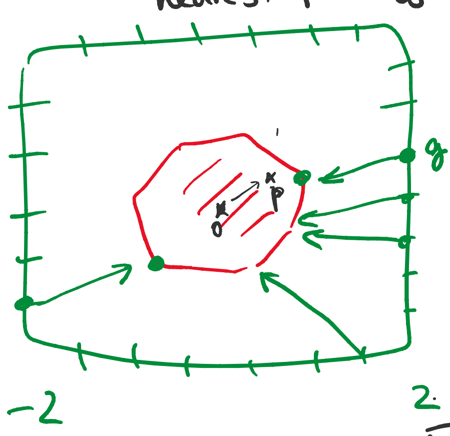


$$
\begin{aligned}
w_u(S) &\ge (p_v - q_v)\cdot u \\
&\ge (p_v - q_v)\cdot v - O(\delta) \\
&\ge (p-q)\cdot v - O(\delta) \\
&\ge (p-q)\cdot u - O(\delta) \\
&= w_u(P) - O(\delta) \\
&\ge (1-O(\delta))\, w_u(P) \quad \text{by fatness}
\end{aligned}
$$

Set $\delta = \Theta(\varepsilon)$.

ε-Kernel Alg'm 3: Dudley's technique '74 for polytope approx.

1. make P fat in $[-1,1]^d$
2. form grid G_δ on bdry of $[-2,2]^d$ with side length δ $\boxed{\text{with } \delta = \Theta(\sqrt{\varepsilon})}$
3. for each grid pt $g \in G_\delta$, put nearest pt $p_g \in P$ to g, nearest pt $q_g \in P$ to $-g$ add to S.



-2 $\qquad$ 2

$\Rightarrow$ size $O\left(\frac{1}{\delta^{d-1}}\right) = \boxed{O\left(\frac{1}{\varepsilon^{\frac{d-1}{2}}}\right)}$.

runtime $\boxed{O\left(\frac{1}{\varepsilon^{(d-1)/2}} n\right)}$

Correctness: $\forall$ unit vector v,

Say $w_v(P) = (p-q)\cdot v$

let $g \in G_\delta$ st. $\angle(g-p, v) \le \delta$.

$x \cdot y \le \|x\|\|y\|$

$x \cdot y = \frac{\|x\|\|y\|}{\cos\delta}$

$$\begin{aligned}
p_g \cdot v &= g\cdot v - (g - p_g)\cdot v \\
&\geq g\cdot v - \|g-p_g\|\,\|v\| \\
&\geq g\cdot v - \|g-p\|\,\|v\| \\
&\geq g\cdot v - \frac{(g-p)\cdot v}{\cos\delta} \\
&= g\cdot v - \frac{(g-p)\cdot v}{1-\Theta(\delta^2)} \\
&= g\cdot v - (g-p)\cdot v \pm \Theta(\delta^2) \\
&\geq p\cdot v - O(\delta^2)
\end{aligned}$$

Similarly,

$$-q_g\cdot v \geq -q\cdot v - O(\delta^2)$$

$$\begin{aligned}
\Rightarrow \quad w_u(S) &\geq (p_g - q_g)\cdot v \\
&\geq (p-q)\cdot v - O(\delta^2) \\
&\geq (1 - O(\delta^2))\, w_u(P) \quad \text{by fatness}
\end{aligned}$$

Set $\delta = \Theta(\sqrt{\varepsilon})$. $\square$

Improving runtime:

combine!

$$O\left(n + \frac{1}{\varepsilon^{(d-1)/2}} \cdot \frac{1}{\varepsilon^{d-1}} \right)$$

$$= \boxed{O\left(n + \frac{1}{\varepsilon^{3(d-1)/2}} \right)} \text{ time}$$

C'04: recurse in dim $\Rightarrow O\left(n + \frac{1}{\varepsilon^{d}}\right)$ roughly.

$\downarrow$

C'17
Arya et al. '17 $O\left(n + \frac{1}{\varepsilon^{d/2}}\right)$ roughly