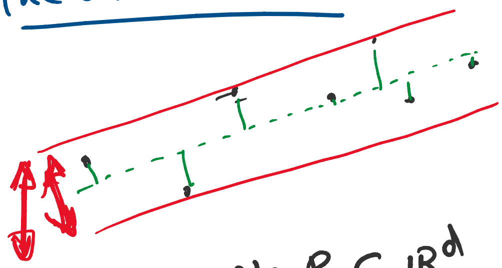
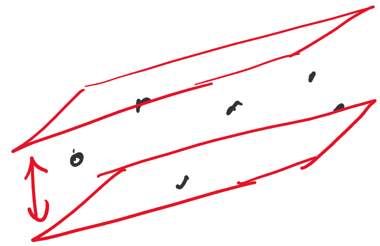
The Width Problem



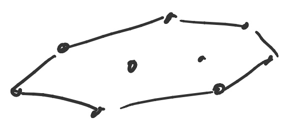


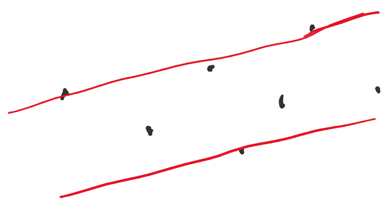
Given n pts $P \subseteq \mathbb{R}^d$,
find 2 parallel hyperplanes enclosing P
minimizing dist. w^*

(appl. line fitting in 2D, hyperplane fitting)

Known exact alg'ms:

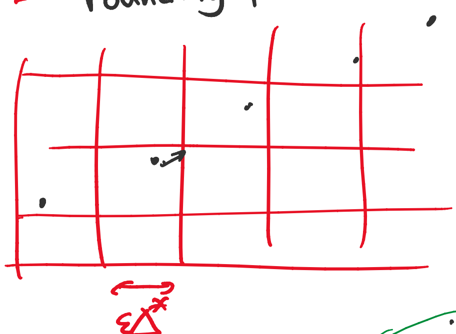
$d=2$	$O(n \log n)$
$d=3$	$O^*(n^{3/2})$
$d \geq 4$	$O(n^{\lceil d/2 \rceil})$





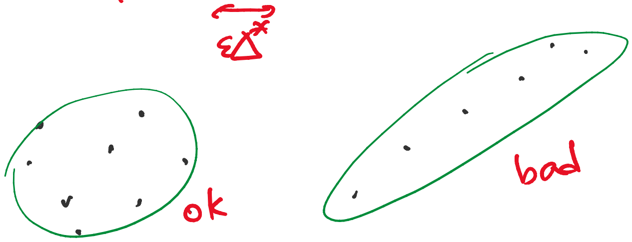
Approx alg'ms?

ideas - rounding pts



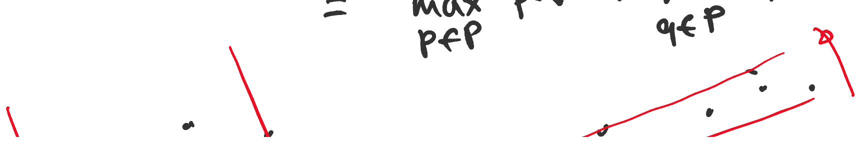
not work!

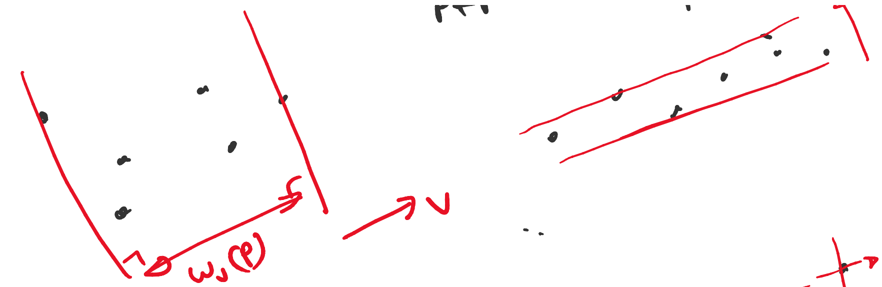
error $O(\varepsilon\Delta^*)$, would be much larger than εw^*



- rounding dirs

given unit vector v,
define $w_v(P)$ = width along dir v
$= \max_{p \in P} p \cdot v - \min_{q \in P} q \cdot v$

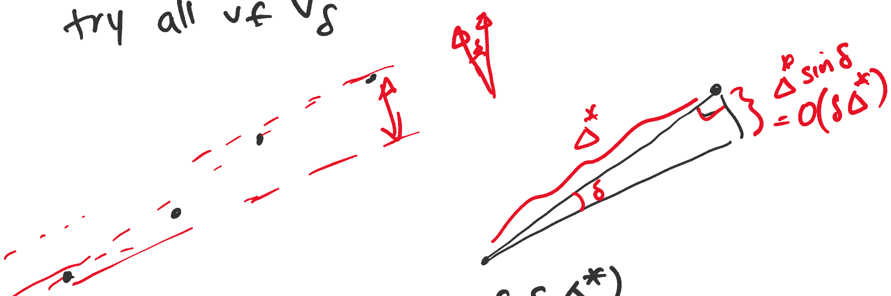




want $w^* = \min_{\text{unit } v} w_v(P)$

(diameter $\Delta^* = \max_{\text{unit } v} w_v(P)$)

form set V_δ of $O(1/\delta^{d-1})$ dirs
try all $v \in V_\delta$



$\Rightarrow$ additive error $O(\delta\Delta^*)$
not $O(\delta w^*)$

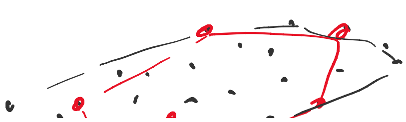
not work!

Duncan-Goodrich-Ramos '97:
$(1+\varepsilon)$-approx. in $O\left(\left(\frac{1}{\varepsilon}\right)^{\frac{d-1}{2}} n\right)$ time

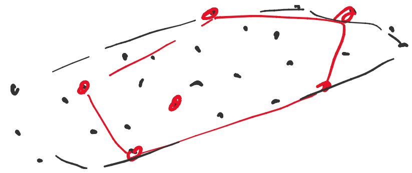
(more clever way of rounding dirs)

→ Agarwal-Har-Peled-Varadarajan '04:
Solve more general problem
of approximating width along all dirs

Def An ε-kernel (also called ε-coreset for directional width)
is a subset $S \subseteq P$ s.t.
$\forall$ dir $v \in \mathbb{R}^d$, $w_v(P) \geq \boxed{w_v(S) \geq (1-\varepsilon)w_v(P)}$



related to
"convex hull"



related to "approx. convex hull"

1. compute an ε-kernel S $\quad \frac{1}{1-\varepsilon} \sim 1+\varepsilon$
2. return $(1+\varepsilon)$-approx of width of S $\quad$ time $O\left(\left(\frac{1}{\varepsilon^{(d-1)/2}}\right)^{\lceil d/2 \rceil}\right)$
   or $O\left(\frac{1}{\varepsilon^{(d-1)/2}} \cdot \frac{1}{\varepsilon^{(d-1)/2}}\right) = O\left(\frac{1}{\varepsilon^{d-1}}\right)$

Main Thm $\quad \exists$ ε-kernel of size $O\left(\frac{1}{\varepsilon^{(d-1)/2}}\right)$

tight

How to find ε-kernel?

Def $\quad P$ is *fat* if (min) width of $P \geq$ const $\cdot$ diameter

Lemma 1 $\quad \exists$ affine transformation that makes P fat.

Lemma 2 $\quad$ Let M be an affine transform.
If $M(S)$ is ε-kernel of $M(P)$,
then S is ε-kernel of P.

Pf. of Lem2:

$$(Mp) \cdot v = (Mp)^T \cdot v$$
$$= p^T M^T v$$

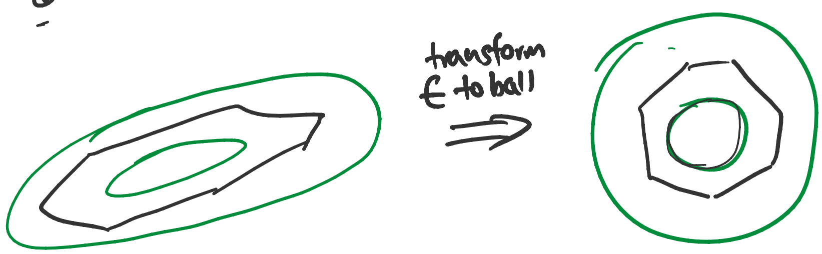
$$= p \cdot M v$$
$$= p \cdot (M^T v). \quad \square$$

Pf of Lem 1:

John Ellipsoid (1948)

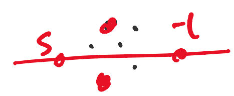
$\exists$ ellipsoid E s.t.

$$\tfrac{1}{d} E \subseteq CH(P) \subseteq E$$



Pf 2 of Lem 1: (Barequet - Har-Peled '97)

$\text{transform}_d(P)$:



1. s = any pt of P
2. t = farthest pt from s
3. rotate & scale s.t. $s = (0, 0, \ldots, 0)$, $t = (1, 0, \ldots, 0)$
4. project each $p \in P$ to $\hat{p}$ along 1st coord.
5. $\text{transform}_{d-1}(\{\hat{p} : p \in P\})$.

$O(n)$ time

Justification: TO BE CONT'D ...