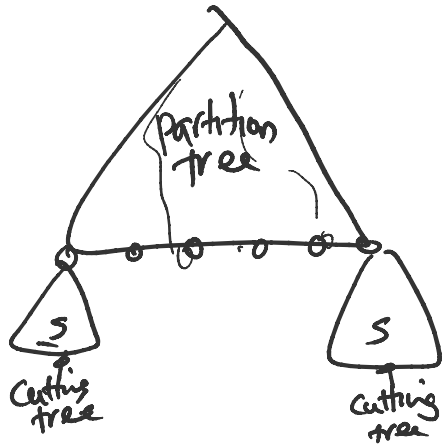
$= O(\log m + \sqrt{t})$ ↑ $O(t^{1-1/d})$ □

Rmk - if t is const, $O(n)$ time.

Consequences - 1. space/time tradeoff



build partition tree when # pts $\le s$, build cutting tree

$\Rightarrow$ space $O\left(n + \frac{n}{s} \cdot s^{2+\varepsilon}\right)$

$= O(n s^{1+\varepsilon})$

$\approx O(m)$ (set $s \approx \frac{m}{n}$)

hide n^{ε} factors

preproc time

query time $O\left(\left(\frac{n}{s}\right)^{\frac{1}{2}+\varepsilon} \log s\right)$

$\approx \boxed{O^*\left(\frac{n}{\sqrt{m}}\right)}$ $\propto \frac{n}{m^{1/d}}$

2. Want to answer n queries for n pts in 2D

$$O^*\left(\underbrace{m}_{\text{preproc time}} + \underbrace{n \cdot \frac{n}{\sqrt{m}}}_{\text{query time}}\right)$$

$m = \frac{n^2}{\sqrt{m}}$

$m^{3/2} \ge n^2$

Set $m = n^{4/3}$

$= \boxed{O^*\left(n^{4/3}\right)}$ ←

3 related to Szemeredi-Trotter Thm '83 ←

3. related to Szemeredi-Trotter Thm '83 ←
in combinatorial geometry:

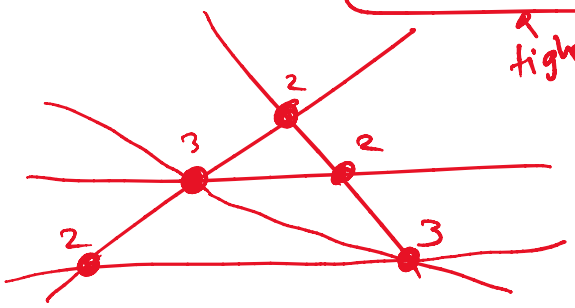
given n lines & n pts in $\mathbb{R}^2$,

pairs (p, ℓ) with p incident on ℓ

is $O(n^{4/3})$

tight!

algorithmic version → "Hopcroft's Problem"



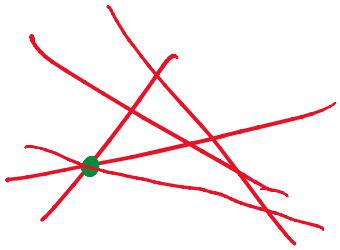
5 lines
5 pts
12 incidences

Pf: $I(n,m)$ = max # incidences for n lines, m pts

Then $I(n,m) = O(n^2 + m)$

pts on 1 line

Sum of degrees of vertices in arrangement



First Attempt: divide lines into r groups of n/r lines each

$$I(n,n) = O(r \cdot I(\tfrac{n}{r}, n))$$
$$= O(r \cdot ((\tfrac{n}{r})^2 + n))$$

Set $r = \sqrt{n}$

$$= O(n^{3/2})$$

Apply Cutting Lemma

$\Rightarrow$ $O(r^2)$ cells each intersected by $\leq n/r$ lines

Subdivide cells s.t. each has $\leq \frac{m}{r^2}$ pts

$\Rightarrow$ still $O(r^2)$ cells

Then $I(n,n) = O(r^2) \cdot I\left(\frac{n}{r}, \frac{n}{r^2}\right)$

$= O(r^2) \cdot I\left(\frac{n}{r^2}, \frac{n}{r}\right)$ by duality

$= O(r^2)\left[\left(\frac{n}{r^2}\right)^2 + \frac{n}{r}\right]$

$= O(r^2)\left(\frac{n^2}{r^4} + \frac{n}{r}\right)$

$\frac{n^2}{r^4} = \frac{n}{r}$

$n = r^3$

Set $r = n^{1/3}$

$= \boxed{O(n^{4/3})}$ □

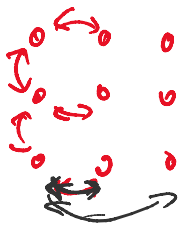
4. also related to Erdös unit distance problem:

given n pts in $\mathbb{R}^2$,

pairs of distance exactly 1

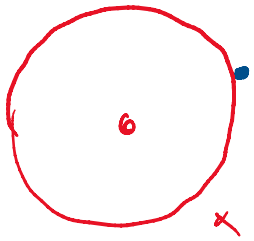
is $O(n^{4/3})$

still open!



Current lower bd:

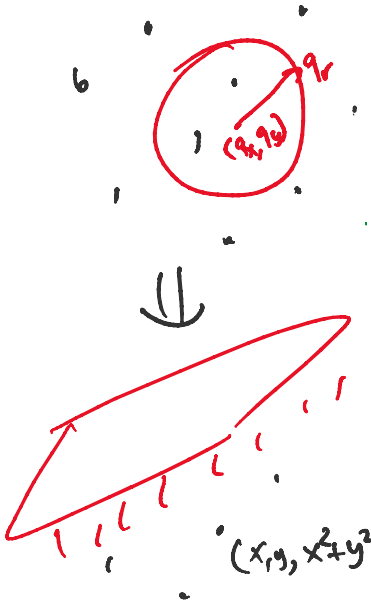
$\Omega\left(n^{1+\frac{c}{\log\log n}}\right)$.



$\Rightarrow$ reduces to incidences between pts & unit circles

5. nonlinear range searching by linearization

e.g. count # pts inside query circle in 2D



find $(x,y) \in P$ s.t.

$$\sqrt{(x-q_x)^2 + (y-q_y)^2} \le q_r$$

i.e. $x^2+y^2 - 2q_x x - 2q_y y + q_x^2 + q_y^2 - q_r^2 \le 0$

z

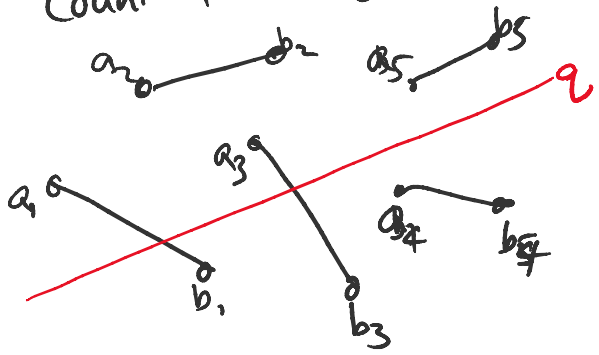
$$z - 2q_x x - 2q_y y + q_x^2 + q_y^2 - q_r^2 \le 0$$

linear in x, y, z.

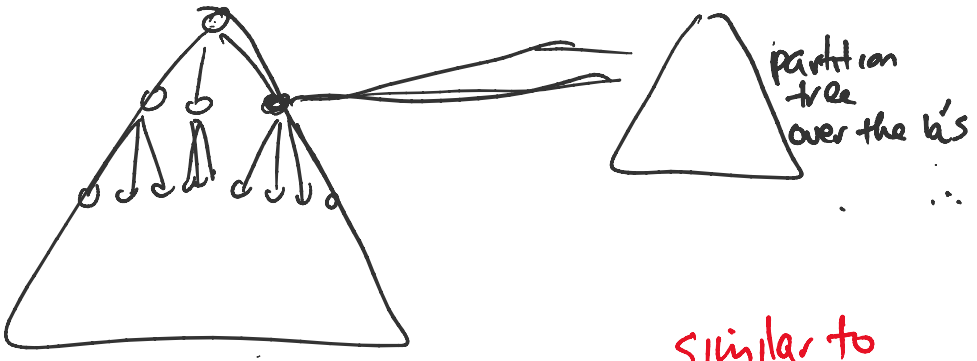
reduces to 3D halfspace range counting

6. multilevel partition trees

e.g. count # line segments intersecting query line in 2D



want all i with a_i above q AND b_i below q



similar to range tree

partition tree over the a_i's

similar to range tree

$$S(n) = t\,S\left(\frac{n}{t}\right) + \cancel{O(t\,S_2(n))} \quad O(n) \quad + \text{const}$$

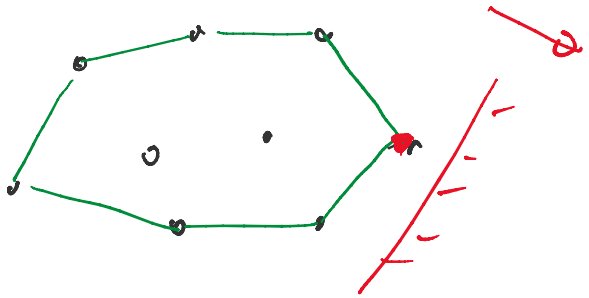
$$\Rightarrow \quad O(n \log n)$$

$$Q(n) = O(\sqrt{t})\,Q\left(\frac{n}{t}\right) + \cancel{O(t\,Q_2(n))} \quad O^*(\sqrt{n})$$

$$\Rightarrow \quad O^*(\sqrt{n})$$

$\vdots$

What about halfspace range reporting?



2D emptiness? $O(n)$ space, $O(\log n)$ time by binary search

3D emptiness?