Rmk: if $t$ is const, $O(n)$ time.

Consequences: 1. Space/time tradeoff

Query time

$$O\left( \left( \frac{\log s}{s} \right)^{t+\varepsilon} \log s \right)$$

$$\approx O\left( \frac{n}{\sqrt{m}} \right)$$

2. Want to answer $n$ queries for $n$ pts in 2D

$$O^{*}\left( m + n \cdot \frac{n}{\sqrt{m}} \right)$$

Set $m = n^{4/3}$

$$= O^{*}\left( \frac{n^{4/3}}{n^{1/2}} \right)$$

3. Related to Szemerédi-Trotter Theorem '83
3. Related to Szemerédi-Trotter Thm '83

in combinatorial geometry:

given \( n \) lines \( \& \) \( n \) pts in \( \mathbb{R}^2 \),

\[ \# \text{ pairs } (p, q) \text{ with } p \text{ incident on } l \]

is \( \mathcal{O}(n^{4/3}) \)

Algorithmic version

"Hopcroft's Problem"

\[ \text{5 lines} \]
\[ \text{5 pts} \]
\[ 12 \text{ incidences} \]

\[ \text{Pf: } I(n, m) = \max \# \text{ incidences for } n \text{ lines, } m \text{ pts} \]

Then \( I(n, m) = \mathcal{O}(n^2 + m) \)

First attempt: divide lines into \( r \) groups of \( \frac{n}{r} \) lines each

\[ I(n, n) = \mathcal{O}(r \cdot I(\frac{n}{r}, \frac{n}{r})) = \mathcal{O}(r \cdot (\frac{n^2}{r^2} + n)) = \mathcal{O}(n^{3/2}) \]

Set \( r = \sqrt{n} \)

Apply Cutting Lemma

\( \Rightarrow \) \( \mathcal{O}(r^2) \) cells each intersected by \( \leq n/r \) lines
Subdivide cells s.t. each has $\leq \frac{n}{r^2}$ pts

\[ \Rightarrow \text{ still } O(r^2) \text{ cells} \]

Then $I(n, n) = O(r^2) \cdot I\left(\frac{n}{r}, \frac{n}{r}\right)$

by duality

\[ = O(r^2) \left( \frac{n^2}{r^2} + \frac{n}{r^2} \right) \]

\[ = O(r^2) \left( \frac{n^2}{r^4} + \frac{n}{r^2} \right) \]

Set $r = n^{\frac{1}{3}}$ = $O\left(n^{\frac{4}{3}}\right)$

4. also related to Erdős unit distance problem:

given $n$ pts in $\mathbb{R}^2$,

# pairs of distance exactly 1 unit is $O\left(n^{\frac{4}{3}}\right)$

still open!

Current lower bd:

\[ \Omega\left(n \left(1 + \frac{c}{\log \log n}\right)\right) \]

implies reduces to incidences between pts & unit circles
5. **nonlinear range searching**
   by linearization

   e.g. count # pts inside query circle in 2D

   \[
   \text{find } (x, y) \in P \text{ s.t.} \quad \sqrt{(x-q_x)^2 + (y-q_y)^2} \leq q_r \\
   \text{i.e. } x^2 + y^2 - 2q_x x - 2q_y y + q_x^2 + q_y^2 - q_r^2 \leq 0 \\
   \text{linear in } x, y, z. \\
   \text{reduces to 3D halfspace range counting}
   \]

6. **multilevel partition trees**

   e.g. count # line segments intersecting query line in 2D

   \[
   \text{want all } a_i \text{ with } \overline{a_i} \text{ above } a \text{ and } b_i \text{ below } a \\
   \text{partition tree over the } b_i's \\
   \text{similar to range tree}
   \]
Partition tree over the axis

\[ S(n) = t S\left(\frac{n}{t}\right) + O(t S_2(n)) + \text{const} \]

\[ \Rightarrow \quad O(n \log n) \quad \text{by binary search} \]

\[ Q(n) = O(V + E) \cdot O(\frac{n}{t}) + O(t S_2(n)) \]

\[ \Rightarrow \quad O^*(Vn) \]

What about halfspace range reporting?

2D emptiness?

O(n) space, O(log n) time by binary search

3D emptiness?