Triangle Range Search

Cutting Lemma: Given \( n \) lines \( L \) in \( \mathbb{R}^2 \), can cut \( \mathbb{R}^2 \) into \( O(r^2) \) cells such that each cell intersects \( \leq \frac{n}{r} \) lines.

\[ \Rightarrow O(n^{d+\varepsilon}) \text{ space, } O(\log n) \text{ time} \]

(Matrak '81)

Partition Thm: Given \( n \) pts \( P \) in \( \mathbb{R}^2 \), can partition into \( t \) subsets \( P_1, \ldots, P_t \) each with \( \approx \frac{n}{t} \) pts & find \( t \) cells \( \Delta_1, \ldots, \Delta_t \) with \( P_i \subseteq \Delta_i \)

such that any line crosses \( O(V_{\varepsilon}) \) cells or \( O(t^{1-\varepsilon}) \)

\[ \Rightarrow O(n) \text{ space, } O(\sqrt{n} \log \frac{O(n)}{n}) \text{ time} \]

\( n^{1-\varepsilon} \)

Proof of Partition Thm:

Suffices to prove crossing \( \leq m \) for a finite set \( L \) of "test" lines.

\( m = O(n^2) \) not difficult.

By considering lines thru 2 pts with more work, can reduce \( m = O(\varepsilon) \)
(with more work, can reduce \( m = O(\varepsilon) \))

**First Attempt:**

1. Apply Cutting Lemma to \( L \) with \( r = V \)
   \[
   \Rightarrow \text{ # cells } = O(r) = O(\varepsilon) \quad \text{ }^{\text{1/2}}
   \]
2. Subdivide cells to ensure each with \( \leq \frac{t}{V} \) pts
   \[
   \Rightarrow \leq t \text{ vertical cuts}
   \Rightarrow \text{ # cells remains } O(t)
   \]

**Analysis:** total crossings between lines & cells

\[
O(t \cdot \frac{m}{V}) = O(m \cdot \varepsilon)
\]

\[
\Rightarrow \text{ average # crossings per line } = O(V \cdot \varepsilon)
\]

**But how to turn average to max?**

**Idea:** multiplicative weight update (Welzl '88)

**Iterative reweighting**

**Matoušek's Alg'm:**

- Define multiset \( \hat{L} \), initialize to \( L \), all multiplicities = 1
  - "weight" / "importance"

for \( i = t, \ldots, 1 \) do:

  // assume \( \frac{m}{V} \) pts remaining

  1. Apply Cutting Lemma to \( \hat{L} \) with \( r = \sqrt{t} \)
     \[
     \Rightarrow \text{ # cells } = O(r^2) \approx i
     \]
  2. Pick cell \( \triangle \) with \( \geq \frac{t}{V} \) pts
  3. Shrink \( \triangle \) to have exactly \( \frac{t}{V} \) pts \( P_i \),

\( \triangle \)
3. shrink $\Delta_i$ to have exactly $\frac{n}{k}$ pts $P_c$.
4. for each $\ell \in L$ crossing $\Delta_i$,
double multiplicity of $\ell$ in $\hat{\ell}$

$e_i = 17$

Analysis: before iteration $i$,

let $m_i = |\hat{\ell}|$ (multiplicity included)

$e_i = |\{ \ell \in \hat{\ell} : \ell \text{ crosses } \Delta_i \}|$

Know $e_i \leq \frac{m_i}{r_i} \approx \frac{m_i}{\sqrt{i}}$.
by Cutting Lem.

$\Rightarrow m_{i+1} \leq m_i + e_i \leq \left(1 + \frac{1}{r_i}\right) m_i$

$\Rightarrow m_{\text{final}} = m_0 \leq \left(1 + \frac{1}{r_1}\right) \left(1 + \frac{1}{r_2}\right) \cdots \left(1 + \frac{1}{r_i}\right) m$

$= m \prod_{i=1}^{t} \left(1 + \frac{1}{r_i}\right)$

$\leq m \prod_{i=1}^{t} e^{1/r_i}$

$\leq m e^{\sum_{i=1}^{t} \frac{1}{r_i}} \approx m e^{O(1/k)}$

$\frac{e^{t}}{e^{r}} \approx O(e^{r})$

But $m_{\text{final}} = \sum_{\ell \in L} (\text{final multiplicity of } \ell)$

$= \sum_{\ell \in L} 2$ (Crossing # of $\ell$)
\[ \forall \mathcal{L}, \quad \text{crossing # of } \mathcal{L} \leq \log_2 m + \sqrt{m} \]

\[ \mathcal{O}(\log m + \sqrt{m}) \]

**Remark** - if \( t \) is const, \( \mathcal{O}(n) \) time.

**Consequences** - space/time tradeoff

- build partition tree
  - when \( t \) pts \( \leq s \),
  - build cutting tree

\[ \Rightarrow \text{space } \mathcal{O}(n + \frac{n}{s} \cdot s^{2+\varepsilon}) \]

\[ = \mathcal{O}(ns^{1+\varepsilon}) \]

\[ \approx \mathcal{O}(m) \quad \text{for } s = \frac{m}{n} \]

query time:

\[ \mathcal{O}\left(\left(\frac{n}{s}\right)^{1+\varepsilon} \log s\right) \]

\[ \approx \mathcal{O}\left(\frac{n}{\sqrt{m}}\right) \]