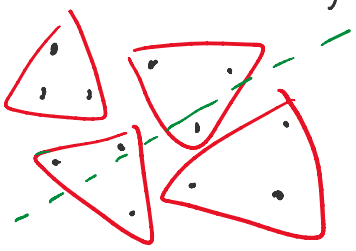
Triangle Range Search

Cutting Lemma (Clarkson '87) Given n lines L in $\mathbb{R}^2$, can cut $\mathbb{R}^2$ into $O(r^2)$ disj. cells s.t. each cell intersects $\leq \frac{n}{r}$ lines. ← $O(r^d)$

$\Rightarrow$ $O(n^{d+\varepsilon})$ space, $O(\log n)$ time

Partition Thm (Matoušek '91) Given n pts P in $\mathbb{R}^2$, can partition into t subsets $P_1, \ldots, P_t$ each with $\sim \frac{n}{t}$ pts & find t cells $\Delta_1, \ldots, \Delta_t$ with $P_i \subseteq \Delta_i$ s.t. any line crosses $O(\sqrt{t})$ cells ← $O(t^{1-1/d})$



$\Rightarrow$ $O(n)$ space

$O(\sqrt{n}\log^{O(1)} n)$ time

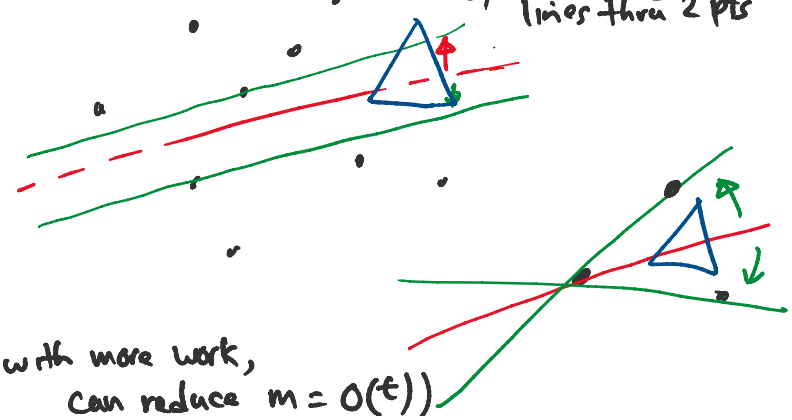
$n^{1-1/d}$

Proof of Partition Thm:

{ Suffices to prove crossing # for a finite set L of "test" lines

($m = O(n^2)$ not difficult) $\hat{m}$

& by considering lines thru 2 pts

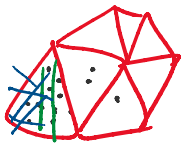


(with more work, can reduce $m = O(t)$)

(with more work, can reduce $m = O(t)$)

First Attempt:

1. Apply Cutting Lemma to L with $r \approx \sqrt{t}$
   $\Rightarrow$ # cells $= O(r^2) \approx O(t)$ (r^2: d; t: $t^{1/d}$)
2. subdivide cells to ensure each with $\le \frac{n}{t}$ pts

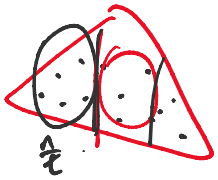


$\Rightarrow \le t$ extra vertical cuts
$\Rightarrow$ # cells remains $O(t)$

Analysis: total crossings between lines & cells

$$O\left(t \cdot \frac{m}{r}\right) = O(m\sqrt{t})$$

(t: # cells; $\frac{m}{r}$: # lines intersecting each cell; t: $t^{1-1/d}$)



$\Rightarrow$ average # crossings per line $= O(\sqrt{t})$

BUT how to turn average to max?

idea - multiplicative weight update / (Welzl '88)
iterative reweighting

Matoušek's Alg'm:

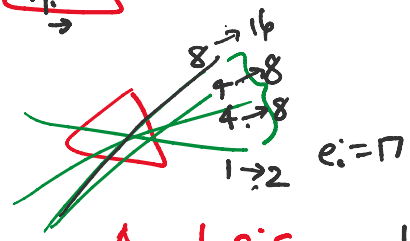
define multiset $\hat{L}$, initialize to L, all multiplicities $= 1$
("weight" / "importance")

for $i = t, \ldots, 1$ do:
// assume $\frac{in}{t}$ pts remaining

1. apply Cutting Lemma to $\hat{L}$ with $r_i \approx \sqrt{i}$
   $\Rightarrow$ # cells $= O(r_i^2) \approx i$
2. pick cell Δ_i with $\ge \frac{n}{t}$ pts
3. shrink Δ_i to have exactly $\frac{n}{t}$ pts P_c,
   remove P_c

3. shrink Δ_i to have exactly $\frac{n}{t}$ pts P_c, & remove P_i

4. for each $\ell \in L$ crossing Δ_i,
double multiplicity of ℓ in $\hat{L}$
e s.t. ℓ won't cross too many future cells



Analysis: before iteration i,

let $m_i = |\hat{L}|$ (multiplicity included)

$e_i = |\{\ell \in \hat{L} : \ell \text{ crosses } \Delta_i\}|$

Know $e_i \le \frac{m_i}{r_i} \approx \frac{m_i}{\sqrt{i}}$.

by Cutting Lem.

$$\Rightarrow m_{i+1} \le m_i + e_i \lesssim \left(1 + \frac{1}{\sqrt{i}}\right) m_i$$

$$\Rightarrow m_{final} = m_0 \lesssim \left(1+\frac{1}{\sqrt{t}}\right)\left(1+\frac{1}{\sqrt{t-1}}\right)\cdots\left(1+\frac{1}{\sqrt{1}}\right) m$$

$$= m \prod_{i=1}^{t} \left(1+\frac{1}{\sqrt{i}}\right) \qquad 1+x \le e^x$$

$$\le m \prod_{i=1}^{t} e^{1/\sqrt{i}} \qquad \sum_{i=1}^{t} i^p \approx \Theta(t^{p+1})$$

$$= m e^{\sum_{i=1}^{t} \frac{1}{\sqrt{i}}} \approx m e^{O(\sqrt{t})}$$

But $m_{final} = \sum_{\ell \in L}$ (final multiplicity of ℓ)

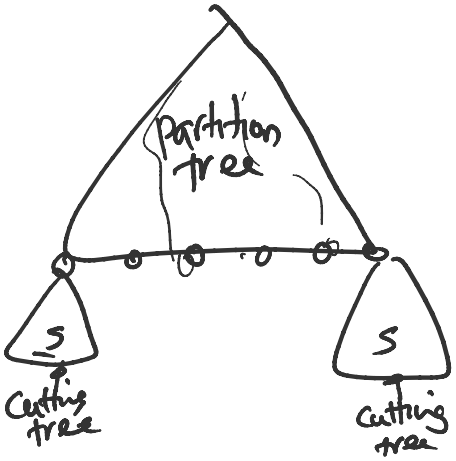
$$= \sum 2^{(\text{crossing \# of } \ell)}$$

$\ell \in L$

$\forall \ell \in L$,
crossing # of $\ell \le \log_2 m_{\text{trial}}$
$= O(\cancel{\log m +} \sqrt{t})$
$\nearrow$
$O(t^{1-1/d})$ □

Rmk - if t is const,
$O(n)$ time.

Consequences - space/time tradeoff



build partition tree
when # pts $\le s$,
build cutting tree

$\Rightarrow$ space $O\left(n + \frac{n}{s} \cdot s^{2+\varepsilon}\right)$
$= O(ns^{1+\varepsilon})$
$\approx \overset{*}{O}(m)$ (set $s \approx \frac{m}{n}$)
hide n^{ε} factors

query time $O\left(\left(\frac{n}{s}\right)^{\frac{1}{2}+\varepsilon} \log s\right)$
$\approx \overset{*}{O}\left(\frac{n}{\sqrt{m}}\right)$