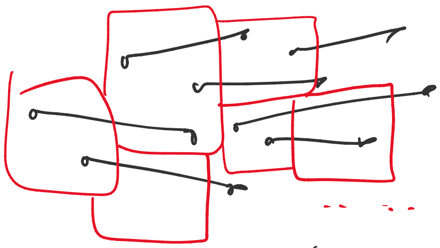
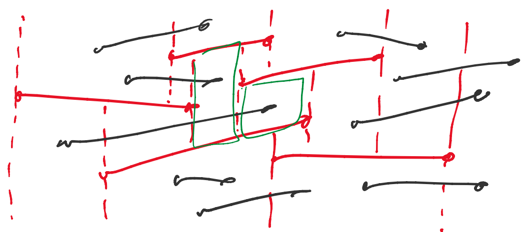
Method 7: Random Sampling (Clarkson-Shor '89)

Cutting Lemma Given n disjoint line segs S, & r, can divide $\mathbb{R}^2$ into $O(r)$ disjoint cells s.t. each cell intersects $O(\frac{n}{r} \log r)$ segs.



Pf:

1. take random sample $R \subseteq S$ of size r
2. return trapezoidal decomposition $T(R)$

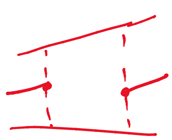


$\Rightarrow O(r)$ cells

$\Pr(\text{alg'm errs}) = \Pr[\text{some cell of } T(R) \text{ intersects } > c\frac{n}{r} \text{ segs}]$

for each cell Δ, let S_Δ = all segs ~~intersecting~~ cross Δ ("conflict list")

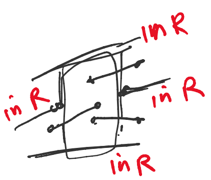
D_Δ : all segs touching $\partial\Delta$ ("defining list")



$|D_\Delta| \leq 4$

Fix trapezoid Δ with $|S_\Delta| > \frac{cn}{r}$, $|D_\Delta| = d$

$\Pr[\Delta \in T(R)] = \Pr[\text{all segs of } D_\Delta \text{ are in } R \text{ \& all segs of } S_\Delta \text{ are not in } R]$



$$= \left(\frac{r}{n}\right)^d \left(1 - \frac{r}{n}\right)^{|S_\Delta|}$$

$$\leq \left(\frac{r}{n}\right)^d \left(1 - \frac{r}{n}\right)^{c\frac{n}{r}}$$

$(1 - x \leq e^{-x})$

$$\leq \left(\frac{r}{n}\right)^d e^{-c}$$

$\Pr(\text{alg'm errs}) \leq$ [illegible] d $(r)^d \cdot e^{-c})$

$$= \left(\frac{r}{n}\right) \cdot e^{-c}$$

$$\Rightarrow \Pr(\text{alg'm errs}) \le O\Big(\sum_{d \le 4} n^d \cdot \left(\frac{r}{n}\right)^d \cdot e^{-c}\Big)$$

$$= O(r^4 \cdot e^{-c}) = O\left(\frac{1}{r}\right) \ll 1.$$

by setting $c = 5 \ln r$. $\square$

Rmk - can remove $\log r$ factor by "re-sampling" in each trapezoid...

Point Location DS:

Apply Cutting Lemma with $r = \frac{n}{b}$

Build DS for $T(R)$ by some known method

recursively build DS in each cell of $T(R)$

$$\Rightarrow \begin{cases} S(n) \le \sum_i S(n_i) + S_0\left(O\left(\frac{n}{b}\right)\right) + O(n) \\ \quad \text{with } \sum_i n_i = O\left(r \cdot \frac{n}{r}\right) = O(n) \quad \text{const factor blow-up} \\ \quad \max_i n_i \le O\left(\frac{n}{r}\right) = O(b). \\ Q(n) \le Q(O(b)) + Q_0\left(O\left(\frac{n}{b}\right)\right) + O(1). \end{cases}$$

$O\left(\frac{n}{\sqrt{b}}\right)$

like before $\Rightarrow$ $\boxed{O(n)}$ space

$\boxed{O(\log n)}$ query time

($O(n \log n)$ preproc time rand.)

Method 8: Randomized Incremental Construction

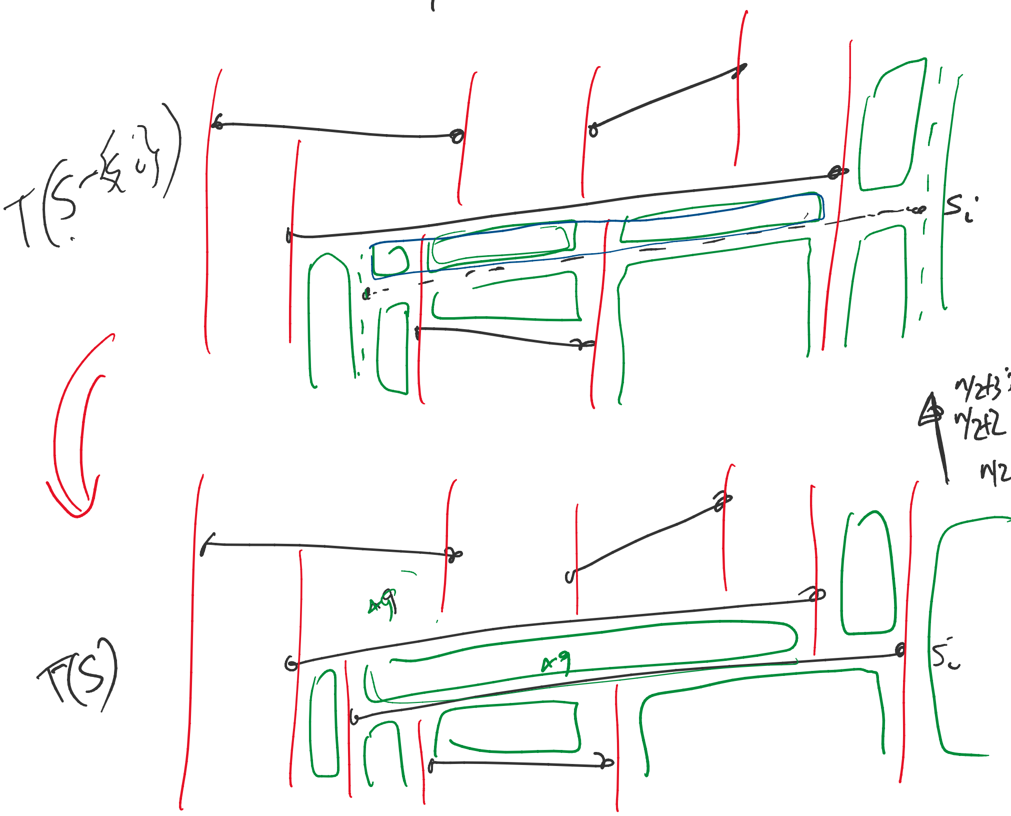
(Clarkson-Shor '89 / Mulmuley '90)

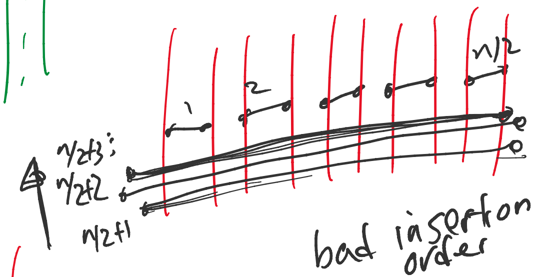
//"backwards version"

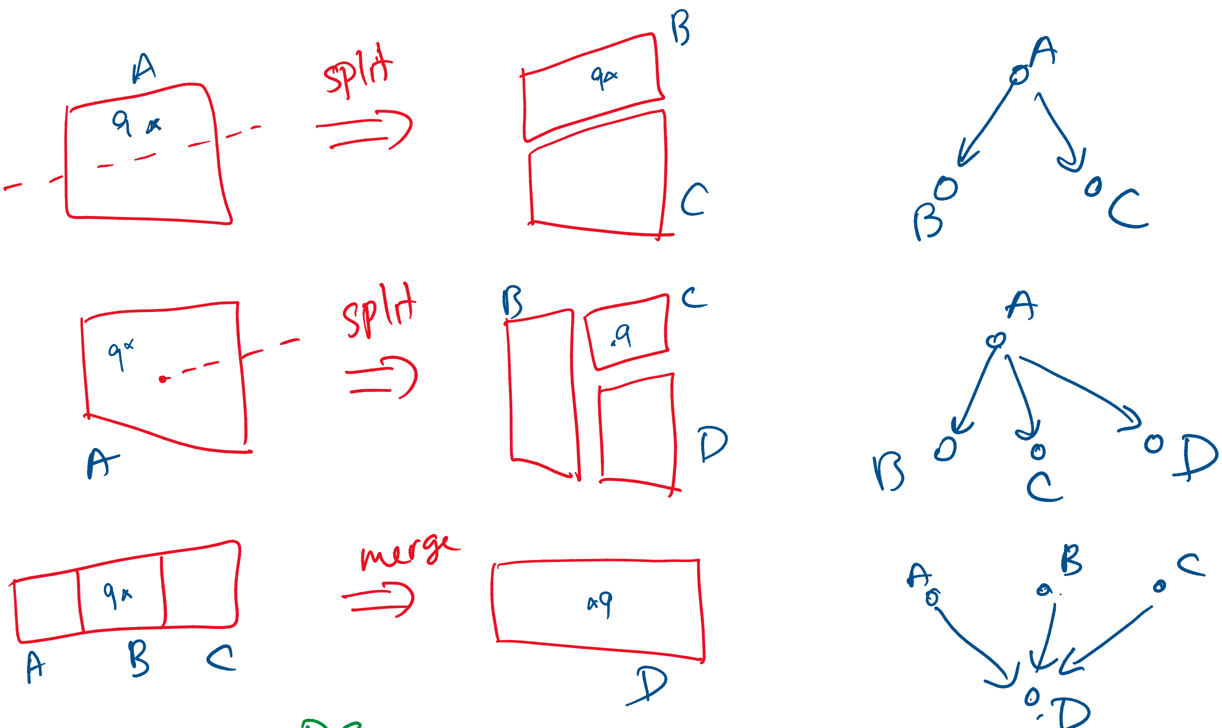
To compute trapezoidal decomposition $T(S)$:

1. pick rand $s_i \in S$
2. compute $T(\underbrace{S - \{s_i\}}_{\text{rand subset of size } n-1})$ recursively

1.
2. compute $\underbrace{T(S-\{s_i\})}_{\text{rand subset of size } n-1}$ recursively
3. add s_i
   by splitting trapezoids & merging trapezoids







Point Location DS:
remember history DAG

Query alg'm:

follow ptrs

Analysis:

Space: line 3:

$\#$ trapezoids created $=$ $\#$ trapezoids $\Delta \in \underline{\underline{T}}(S)$ $\underbrace{\text{s.t. } s_i \text{ touches } \partial\Delta}_{\text{call this } \deg(s_i)}$

$$\sum_{s_i \in S} \deg(s_i) \leq 4n$$

$$\Rightarrow \quad E[\deg(s_i)] \leq 4.$$

(for rand $s_i \in S$)

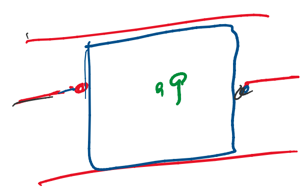
$$\Rightarrow \quad S(n) = S(n-1) + O(1)$$

$$\Rightarrow \quad S(n) = \boxed{O(n)}$$

Query time:

assume query pt q is indep. of rand. choices

let Δ be trapezoid of $T(S)$ containing q



$$\Pr[\text{ need to follow ptr}]$$

$$= \Pr(s_i \text{ touches } \partial\Delta)$$

$$\leq \frac{4}{n}$$

$$\Rightarrow \quad Q(n) \leq Q(n-1) + O\left(\frac{1}{n}\right)$$

$$\Rightarrow \quad O\left(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \dots + 1\right)$$

Harmonic number

$$= \boxed{O(\log n)}$$